

CHROMATIC NUMBERS OF SIMPLICIAL MANIFOLDS

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ABSTRACT. Higher chromatic numbers of simplicial complexes naturally generalize the chromatic number of a graph. Yet, little is known on higher chromatic numbers for specific simplicial complexes.

The 2-chromatic number of any fixed surface is finite. However, asymptotically the 2-chromatic number of surfaces becomes arbitrarily large with growing genus (as we will see via Steiner triple systems). We show that orientable surfaces of genus at least 20 and non-orientable surfaces of genus at least 26 have a 2-chromatic number of at least 4. Via a projective Steiner triple systems, we construct an explicit triangulation of a non-orientable surface, of genus 2542 and with face vector $f = (127, 8001, 5334)$, that has 2-chromatic number 5 or 6. We also give orientable examples with 2-chromatic numbers 5 and 6.

For 3-dimensional manifolds, an iterated moment curve construction can be used to produce triangulations with arbitrarily large 2-chromatic number [6, 18], but of tremendous size. Via a topological version of the geometric construction of [18], we exhibit the first concrete triangulation of the 3-dimension sphere S^3 , with face vector $f = (167, 1579, 2824, 1412)$, that has 2-chromatic number 5.

1. INTRODUCTION

Let $G = (V, E)$ be a (finite) simple graph with vertex set V and edge set E . The 1-chromatic number $\chi_1(G)$ of G , i.e., the (standard) chromatic number $\chi(G)$ of G , is the number of colors needed to color the vertices of G such that no two adjacent vertices are colored with the same color. For given G , it is NP-hard to compute $\chi_1(G)$ [14, 20]. It is even hard to approximate $\chi_1(G)$ [35], and rather few tools (from algebra, linear algebra, topology) are available to provide lower bounds for $\chi_1(G)$; see the surveys [19, 26] for a discussion. Upper bounds for $\chi_1(G)$ usually are obtained by searching for or constructing explicit colorings of G .

Let V be a finite set, and E be a family of subsets of V . The ordered pair $H = (V, E)$ is called a *hypergraph* with *vertex set* V and *edge set* E . If all edges of H have size 2, the hypergraph H is a graph, while, in general, we can think of H to define a simplicial complex K on the vertex set V with facet list E (not necessarily inclusion-free).

A hypergraph $H = (V, E)$ is *r-uniform* if all edges of H are of size r , thus defining a *pure* simplicial complex K of dimension $r - 1$. Conversely, every pure d -dimensional simplicial complex can be encoded as a $(d + 1)$ -uniform hypergraph.

Colorings of hypergraphs generalize colorings of graphs in various ways; see, for example, [21]. The *s-chromatic number* $\chi_s(H)$ [$\chi_s(K)$] of a hypergraph H [a finite simplicial complex K] is the minimal number of colors needed to color the vertices V so that subsets of the sets in E with at most s elements are allowed to be monochrome, but no $(s + 1)$ -element subset is monochrome.

For $s = 1$, the vertices of every edge of E are required to be colored with distinct colors, and $\chi_1(H)$ is called the *strong chromatic number* of H [alternatively, $\chi_1(K)$ is the (standard) chromatic number $\chi(\text{skel}_1(K))$ of the 1-skeleton of K]. For $s = r - 1$, in the case of r -uniform hypergraphs, $\chi_{r-1}(H)$ is called the *weak chromatic number* of H . We say that a hypergraph H [a simplicial complex K] is *(k, l)-colorable* if the vertices of H [of K] can be colored with k colors so that there are no monochromatic $(l + 1)$ -element sets.

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Definition 1.1. Let M be a triangulable (closed, connected) manifold of dimension d and $1 \leq s < d + 1$. Then

$$\chi_s(M) = \sup\{\chi_s(K) \mid K \text{ is a triangulation of } M\}$$

is the s -chromatic number of M .

The s -chromatic numbers of a manifold form a descending sequence

$$\chi_1(M) \geq \chi_2(M) \geq \cdots \geq \chi_d(M) \geq \chi_{d+1}(M) = 1,$$

where some of these numbers are finite and some are infinite:

- (1) $\chi_1(S^1) = 3$,
- (2) $\chi_1(M^2) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi_E(M^2)}}{2} \right\rfloor$ [2, 28, 29] for M^2 different from the Klein bottle, where $\chi_E(M^2)$ denotes the Euler characteristics of M^2 , and $\chi_1(\text{Klein bottle}) = 6$,
- (3) $\chi_1(M^3) = \infty$ for any 3-manifold M^3 [34],
- (4) $\chi_2(M^3) = \infty$ for any 3-manifold M^3 (Section 6),
- (5) $\chi_s(M^d) = \infty$ when M^d a triangulable d -manifold with $d \geq 3$ and $s \leq \lceil d/2 \rceil$ (Section 6).

Further, we have for surfaces and general 2-dimensional simplicial complexes:

- (6) $\chi_1(\text{arbitrary surface}) = \infty$ [28, 29],
- (7) $\chi_2(\text{arbitrary 2-dimensional simplicial complex}) = \infty$ [10, 18],
- (8) $\chi_2(\text{arbitrary surface}) = \infty$ (Section 2).

Case (1): Let C_n be a cycle of length n . Then $\chi_1(C_n) = \chi(C_n) = 2$ for even n and $\chi_1(C_n) = \chi(C_n) = 3$ for odd n . Thus, for the 1-dimensional sphere S^1 , $\chi_1(S^1) = 3$.

Case (2): We have

$$(1.2) \quad \chi_1(S^2) = 4$$

for the 2-dimensional sphere S^2 by the 4-color theorem [2], while for surfaces $M^2 \neq S^2$ and M^2 different from the Klein bottle, the map color theorem [30] states

$$(1.3) \quad \chi_1(M^2) = \left\lfloor \frac{7 + \sqrt{49 - 24\chi_E(M^2)}}{2} \right\rfloor;$$

where in the case of the Klein bottle,

$$(1.4) \quad \chi_1(\text{Klein bottle}) = 6.$$

Thus, the 1-chromatic numbers of surfaces are completely determined.

Case (3): Walkup [34] proved that all (closed, connected) 3-manifolds have neighborly triangulations and thus have $\chi_1 = \infty$. Alternatively, Case (3), and more generally Case (5) for $s < \lceil d/2 \rceil$, follow (for odd d ; for even d see Section 6) by taking connected sums of the boundaries $\partial CP(m, d+1)$ of neighborly cyclic $(d+1)$ -polytopes $CP(m, d+1)$ on m vertices with some triangulation of M^d .

Cases (4) and (7): Heise, Panagiotou, Pikhurko and Taraz [18] provided an inductive geometric construction via the moment curve (this construction was also found independently by Jan Kynčl and Josef Cibulka as pointed out to us by Martin Tancer) to yield 2-dimensional complexes with arbitrary high 2-chromatic numbers. In Case (4), PL topology [6, Theorem I.2.A] (we are grateful to Karim Adiprasito for reminding us of this result) is used to extend geometrically any linearly embedded 2-dimensional simplicial complex in \mathbb{R}^3 to a triangulation of the 3-dimensional ball B^3 that still contains the original 2-dimensional complex as a subcomplex. If the initial complex has high 2-chromatic number then also the triangulated ball it is contained in. By coning off the boundaries of the resulting balls, we obtain triangulations of S^3 with arbitrary high 2-chromatic numbers.

Case (5) with $s = \lceil d/2 \rceil$ is a generalization of Case (4); see Section 6.

Case (6) follows as an immediate consequence of Case (2).

Case (7): Steiner triple systems yield examples with arbitrary large χ_2 [10], as do the examples of Heise, Panagiotou, Pikhurko and Taraz [18].

Case (8): Although we might expect that neighborly triangulations of surfaces should have high 2-chromatic number, we show in Section 4, via Bose Steiner triple systems, that there is an infinite series of neighborly triangulations of surfaces with $\chi_2 = 3$. In contrast, we show in Section 2 that any Steiner triple system can be extended to a triangulation of a surface (orientable or non-orientable). Thus, Steiner triple systems with high χ_2 give rise to triangulated surfaces with high χ_2 .

In Section 2 we discuss 2-chromatic numbers of surfaces. In particular, we show that $\chi_2(\text{arbitrary surface}) = \infty$ (via embeddings of Steiner triple systems with high χ_2) and provide explicit examples of orientable triangulated surfaces with $\chi_2 = 5$ and $\chi_2 = 6$. Section 3 introduces basic concepts of Steiner triple systems and transversals,

which might also be of independent interest in topological design theory. Section 4 discusses neighborly Bose Steiner triple systems with $\chi_2 = 3$. In Section 5, we use projective Steiner triple systems to construct a triangulation of a closed (non-orientable) surface with $f = (127, 8001, 5334)$ and 2-chromatic number 5 or 6. In Section 6, by combining results of [18] and [6], we show that $\chi_s(M^d) = \infty$ for triangulable d -manifolds M^d with $d \geq 3$ and $s \leq \lceil d/2 \rceil$. In Section 7, we provide a topological variant of the geometric iterated moment curve construction of [18] to obtain a first concrete triangulation with $f = (167, 1579, 2824, 1412)$ of the 3-dimension sphere S^3 with 2-chromatic number equal to 5.

2. THE 2-CHROMATIC NUMBER OF SURFACES

Let M^2 be a surface and K a triangulation of M^2 with $\chi_1(K) \leq \chi_1(M^2)$ and color classes $1, \dots, \chi_1(K)$. If we consecutively merge color classes 1 with 2, 3 with 4, etc., we obtain a 2-coloring of K with $\lceil \frac{\chi_1(K)}{2} \rceil$ colors as no three vertices of a triangle of K get colored with the same color. Thus, for any surface M^2 ,

$$(2.1) \quad \chi_2(M^2) \leq \left\lceil \frac{\chi_1(M^2)}{2} \right\rceil.$$

In any 2-coloring of a triangulation K of M^2 , any individual triangle requires at least two colors, giving the trivial lower bound

$$(2.2) \quad \chi_2(M^2) \geq 2.$$

In the case of the 2-sphere S^2 , by combining the two bounds, $2 \leq \chi_2(S^2) \leq \lceil \frac{\chi_1(S^2)}{2} \rceil = 2$, we obtain

$$(2.3) \quad \chi_2(S^2) = 2.$$

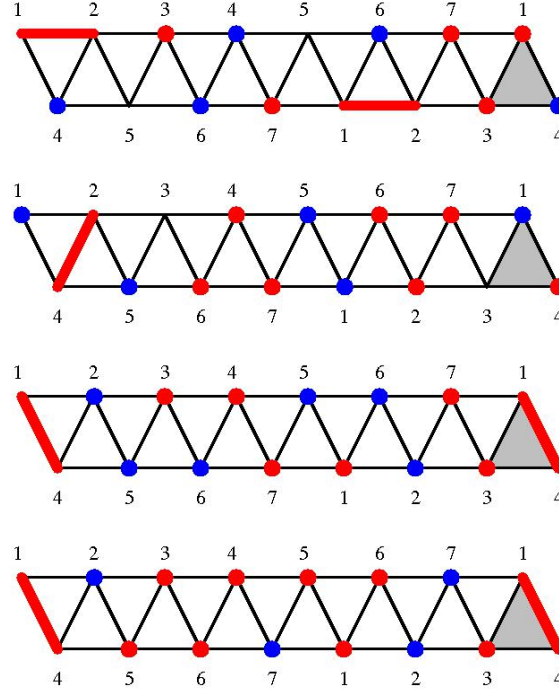


FIGURE 1. The 7-vertex triangulation of the torus with monochromatic triangles in any $(2, 2)$ -coloring.

Theorem 2.4 (Kündgen and Ramamurthi [22]). *For any surface $M^2 \neq S^2$,*

$$(2.5) \quad \chi_2(M^2) \geq 3.$$

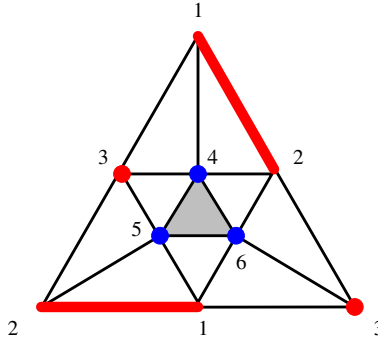
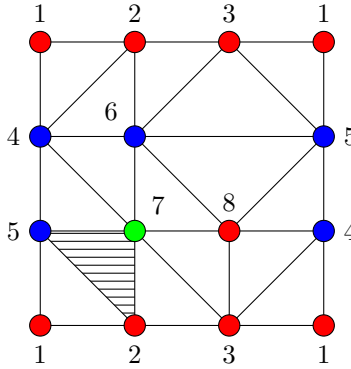


FIGURE 2. The 6-vertex triangulation of the real projective plane with a monochromatic triangle.

FIGURE 3. A $(3, 2)$ -coloring of a combinatorial Klein bottle (minus a triangle).

Proof. Here, we give an alternative proof to [22] to pave the way for Theorem 2.12. We first show that the 2-chromatic number of the unique minimal 7-vertex triangulation of the torus (see Figure 1) is three — and is still three when we remove (up to symmetry) one of the 14 triangles of the triangulation. Let us assume that two colors (red and blue) suffice when we remove triangle 134 (shaded in grey in Figure 1) from the triangulation. We pick a triangle, say 124, and color its three vertices 1, 2 and 4 with two colors such that the triangle is not monochromatic. Thus, two vertices of the triangle get the same color, say, red, while the third vertex is colored differently, say, in blue.

- If 1 and 2 are colored red and 4 is colored blue, it follows that also vertex 6 is colored blue, since otherwise we would have a monochromatic triangle 126. The blue edge 46 then forces the vertices 3 and 7 to be red, which yields monochromatic triangles 137 and 237.
- If 2 and 4 are colored red and 1 is colored blue, also 5 has to be blue. Via the blue edge 15 it then follows that the vertices 6 and 7 are red, which yields monochromatic triangles 267 and 467.
- If 1 and 4 are colored red and 2 is colored blue, the color of vertex 3 remains open, since we removed triangle 134 from the triangulation. We therefore consider two subcases. First, let us assume that 5 is blue. Then 3 has to be red (because of the blue edge 25), 6 has to be blue (because of the red edge 34), and 7 has to be red (because of the blue edge 26), which yields the monochromatic triangle 137. If instead we color 5 red, 7 has to be blue (because of the red edge 45), but then 3 and 6 are red (because of the blue edge 27), which gives monochromatic triangles 156 and 346.

By taking connected sums with the 7-vertex triangulation of the torus (minus a triangle), we have $\chi_2(M^2) \geq 3$ for any surface M^2 with $\chi_E(M^2) < 0$. The only two remaining cases then are the real projective plane \mathbb{RP}^2 (with $\chi_E(\mathbb{RP}^2) = 1$) and the Klein bottle with $\chi_E(\text{Klein bottle}) = 0$.

The vertex-minimal 6-vertex triangulation \mathbb{RP}_6^2 (see Figure 2) of the real projective plane is edge-transitive. Thus, we can choose the edge 12 to be red, which forces the vertices 4 and 5 to be blue and (via the blue edge 45) 3 to be red. Via the red edge 13, 6 has to be blue, which yields a monochromatic triangle 456. If we remove the triangle 456 from the triangulation, we obtain a valid $(2, 2)$ -coloring of the crosscap.

The Klein bottle has six distinct vertex-minimal 8-vertex triangulations, of which three are $(2, 2)$ -colorable, while the other three do not admit a $(2, 2)$ -coloring; see Figure 3 for a 8-vertex triangulation with a $(3, 2)$ -coloring — we leave it to the reader to check that there is no $(2, 2)$ -coloring for this example. \square

Since $\chi_1(\mathbb{RP}^2) = 6$ and $\chi_1(\text{Klein bottle}) = 6$, we have

$$(2.6) \quad \chi_2(\mathbb{RP}^2) = 3$$

and

$$(2.7) \quad \chi_2(\text{Klein bottle}) = 3.$$

Theorem 2.8 (Kündgen and Ramamurthi [22]).

$$(2.9) \quad \chi_2(T^2) = 3.$$

Proof. The unique 7-vertex triangulation of the torus is the only triangulation of the torus with 1-chromatic number 7 [11]; it is 2-colorable with 3 colors. All other triangulations of the torus have 1-chromatic number at most 6 and thus also are 2-colorable with 3 colors. \square

Remark 2.10. The 2-sphere S^2 , the projective plane \mathbb{RP}^2 , the 2-torus T^2 and the Klein bottle are the only surfaces for which their 2-chromatic number are known.

As noted by Kündgen and Ramamurthi [22], there are neighborly triangulations on 19 vertices with $\chi_2 = 4$. We will sharpen this result in Theorem 2.12 for non-orientable surfaces and give an extension to connected sums. For the search for 2-colorings in the case of small values of k we use a simple brute force strategy.

Algorithm 2.11. (Search for $(k, 2)$ -colorings)

INPUT: Let K be a triangulation of a surface M^2 and fix k to search for $(k, 2)$ -colorings.

- (1) Fix a starting triangle of K .
- (2) Fix an initial coloring of the starting triangle. If $k = 2$, we consider the three configurations with a monochromatic edge colored with color 1 and the third vertex colored with color 2. If $k \geq 3$, we also consider the case that the three vertices (in single fixed permutation) of the triangle are colored differently by using the three colors 1, 2, and 3. We write the initial configurations to a stack.

At any stage, we have a subset of colored vertices and the remaining set of unmarked vertices. The unmarked vertices will be split into two sets, the set of *reachable* vertices and the set of *unreachable* vertices. An unmarked vertex is reachable if it forms a triangle (in the triangulation K) together with two colored vertices. A reachable vertex sometimes can be reached via different triangles, which could impose restrictions on the choices to color this vertex to avoid monochromatic triangles. We compute the list of admissible colors for each reachable vertex. If a reachable vertex has an empty list of admissible colors, we discard this configuration.

- (3) We remove a configuration from the stack and choose a reachable vertex for which the cardinality of its list of admissible colors is as small as possible. For any choice to color the new vertex, we write the resulting configuration (if admissible) back to the stack.

At any intermediate stage there always are reachable vertices: let L be the subcomplex of the triangulation K consisting of all triangles for which all of their vertices have been colored. Then for any edge on the boundary of L , one of its two incident triangles is in L while the other is not. The latter triangle contains an uncolored, but reachable vertex.

The search terminates either with the conclusion that K does not have any $(k, 2)$ -colorings or outputs a $(k, 2)$ -coloring.

We use that $\chi_E = 2 - 2g$ for an orientable surface of genus g and $\chi_E = 2 - u$ for a non-orientable surface of genus u .

Theorem 2.12. Let M^2 be an orientable surface of genus $g \geq 20$ (i.e., with $\chi_E(M^2) \leq -38$) or M^2 be a non-orientable surface of genus $u \geq 26$ (i.e., with $\chi_E(M^2) \leq -24$), then

$$(2.13) \quad \chi_2(M^2) \geq 4.$$

Proof. We run the program on various triangulations of surfaces from [25] to find examples that do not admit $(3, 2)$ -colorings. For each such example, we then rerun the search on the example minus one of its triangles:

The example `manifold_cyc_d2_n16_#10` from [25] is a neighborly cyclic triangulation of the non-orientable surface of genus $u = 26$ on $n = 16$ vertices with $\chi_E = -24$, which is $(4, 2)$ -colorable, but not $(3, 2)$ -colorable. If we remove

the triangle $[1, 12, 14]$ from the triangulation, there still is no $(3, 2)$ -coloring. By taking connected sums, we obtain the stated result for non-orientable surfaces with $\chi_E \leq -24$.

The example `manifold_cyc_d2_n19_#47` from [25] is a neighborly cyclic triangulation of the orientable surface of genus $g = 20$ on $n = 19$ vertices with $\chi_E = -38$, which is $(4, 2)$ -colorable, but not $(3, 2)$ -colorable. If we remove the triangle $[6, 8, 15]$ from the triangulation, there still is no $(3, 2)$ -coloring. By taking connected sums, the stated result follows for orientable surfaces with $\chi_E \leq -38$. \square

Problem 2.14. *Is there an orientable surface M of genus $g < 20$ or a non-orientable surface M of genus $u < 26$ with $\chi_2(M^2) \geq 4$?*

We tried bistellar flips [7] to search through the spaces of triangulations for various fixed genus surfaces with $g < 20$ or $u < 26$, but never found an example with $\chi_2 \geq 4$. For $g \geq 20$ or $u \geq 26$ we also failed to find an example with $\chi_2 > 4$ this way. However, as we will see next, there are triangulations of orientable and non-orientable surfaces with arbitrary large 2-chromatic number.

Definition 2.15. *A pure 2-dimensional simplicial complex $\text{STS}(n)$ with vertex set $V = \{1, \dots, n\}$ is a Steiner triple system on V if the 1-skeleton of $\text{STS}(n)$ is the complete graph on V and every 1-simplex lies in a unique 2-simplex of $\text{STS}(n)$.*

For every $k \geq 3$ there is a Steiner triple system with $\chi_2 \geq k$ [10]. However, the proof of existence in [10] is non-constructive, and explicit examples are not known for $k \geq 7$. Steiner triple systems $\text{STS}(n)$ with $n \geq 7$ have $\chi_2 \geq 3$ [32], Steiner triple systems with $\chi_2 = 4$ can be found, for example, in [10, 32].

For projective Steiner triple systems $\text{PG}(2^d)$ (see Section 5), $\chi_2(\text{PG}(2^d)) \leq \chi_2(\text{PG}(2^{d+1})) \leq \chi_2(\text{PG}(2^d)) + 1$ [32], where $\chi_2(\text{PG}(2^5)) = 4$ [32] and $\chi_2(\text{PG}(2^6)) = 5$ [13]. By [17, Corollary 2], $\chi_2(\text{PG}(2^d)) \rightarrow \infty$ for $d \rightarrow \infty$.

Grannell, Griggs, and Širáň [16] proved that for $n > 3$ every $\text{STS}(n)$ has both an orientable and a nonorientable surface embedding in which the triples of the $\text{STS}(n)$ appear as triangular faces and there is just one additional large face.

We combine the existence of Steiner triple systems with high χ_2 with the embedding theorem [16].

Theorem 2.16. $\chi_2(\text{arbitrary surface}) = \infty$.

Proof. For any given $\text{STS}(n)$ on n vertices, Grannell, Griggs, and Širáň [16] construct *maximum genus embeddings* (orientable and non-orientable) that decompose the target surface into the triangles of the $\text{STS}(n)$ plus one additional polygonal cell that has as its boundary the $\binom{n}{2}$ edges (of the complete graph K_n as the 1-skeleton) of the $\text{STS}(n)$. In particular, each vertex of the $\text{STS}(n)$ appears $\binom{n}{2}/n$ times on the boundary of the polygonal cell, but any three consecutive vertices on the boundary differ (as there are no loops and multiple edges).

We extend the embeddings of [16] to proper triangulations by placing a cycle with $\binom{n}{2}$ additional vertices in the interior of the polygonal cell and triangulate the annulus between the inner cycle and the boundary cycle in a zigzag way, which gives $2\binom{n}{2}$ triangles in the annulus. A triangulation by placing chords in the inner disk requires $\binom{n}{2} - 2$ additional triangles. If $\chi_2(\text{STS}(n)) = k$, then the resulting triangulated surfaces (orientable and non-orientable) have face vector

$$(2.17) \quad f = (n + \binom{n}{2}, 5\binom{n}{2} - 3, \frac{1}{3}\binom{n}{2} + 3\binom{n}{2} - 2),$$

$$\chi_E = 2 - \frac{1}{3}(n-1)(n-3) \text{ and } \chi_2 \geq k.$$

The genus of the resulting surface is $\frac{1}{6}(n-1)(n-3)$ in the orientable and $\frac{1}{3}(n-1)(n-3)$ in the non-orientable case. The face vector (2.17) of the triangulations can further be improved by not placing an inner cycle vertex for every edge on the identified boundary of the polygonal cell, but instead have an inner cycle vertex for a sequence of consecutive boundary edges that contain no repeated original vertices. Yet, for the simplicity of the construction, we place all $\binom{n}{2}$ new vertices. \square

Theorem 2.18. *Every Steiner triple system $\text{STS}(n)$ with 2-chromatic number $k \geq 3$ has embeddings into triangulations of the orientable surface of genus $g = \frac{1}{6}(n-1)(n-3)$ and the non-orientable surface of genus $u = \frac{1}{3}(n-1)(n-3)$ with*

$$(2.19) \quad f = (n + \binom{n}{2} + 1, 5\binom{n}{2}, \frac{1}{3}\binom{n}{2} + 3\binom{n}{2})$$

that also have 2-chromatic number k .

Proof. Instead of adding chords to triangulate the inner disc, we place a central vertex in the interior of the inner cycle and add the cone with respect to the cycle. This yields triangulations that have one vertex, three edges and two triangles extra and thus have $f = (n + \binom{n}{2} + 1, 5\binom{n}{2}, \frac{1}{3}\binom{n}{2} + 3\binom{n}{2})$.

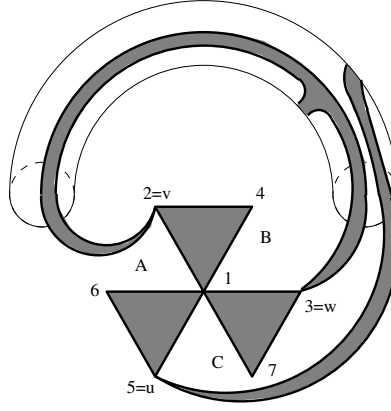


FIGURE 4. The addition of a triangle to the embedded vertex star of STS(7).

Let the n vertices (with repetitions) on the outer cycle be colored with k colors such that the Steiner triple system STS(n) has no monochromatic triangle. Then only two of the colors $1, \dots, k$ are needed to color the inner cycle: For this, we go along the inner cycle of the zigzag collar and color the vertices with color 1 until a neighboring vertex on the outer cycle has color 1 as well. In this case, we switch to color 2 until an outer vertex is colored with color 2, then switch back to color 1, etc. For the cone vertex we are then free to use color 3. \square

The following triangulation scheme provides an algorithmic version of the embedding procedure of [16] along with the completion to a triangulation.

Algorithm 2.20. (Triangulation of an orientable maximum genus embedding of a Steiner triple system)

INPUT: Let STS(n) be a Steiner triple system on n vertices $1, \dots, n$.

- (1) Start with an embedding of the star of the vertex 1 in STS(n) into the 2-sphere S^2 . For this, there are $\frac{1}{2} \binom{n-1}{2} 2^{\frac{n-1}{2}}$ choices.
- (2) Read off the boundary cycle of the embedded star. Let the triangles of the star be ‘black’ and the outside region (a disk with identifications on the boundary) be ‘white’.
- (3) Proceed iteratively for the remaining triangles of the STS(n) (in any order). For each picked triangle uvw , add a handle to the white area. Choose occurrences of the vertices u, v, w on the boundary cycle and glue in the triangle uvw by routing it via the newly added handle in a twisted way (see Figure 4), such that after the addition of uvw , the white outside area is a disk again. For the chosen u, v, w , let $uAvwBwC$ be the boundary cycle before the addition of uvw . By the addition of the handle and uvw , the boundary cycle is then expanded to $uAvwCuvBw$, by inserting the edges vw, uv , and wu and by traversing C before B .
- (4) Once all remaining triangles of STS(n) are added, the boundary cycle contains $\binom{n}{2}$ vertices (multiple copies) and all $\binom{n}{2}$ edges of the complete graph K_n (i.e., the boundary cycle is an Eulerian cycle through K_n).
- (5) Place a second cycle with $\binom{n}{2}$ new vertices in the interior of the white disk and triangulate the annulus between the two cycles in a zigzag way.
- (6) Finally, triangulate the inner polygon, e.g., by adding the cone of an additional vertex as apex with respect to all edges on the inner polygon.

For non-orientable embeddings, Step (3) of Algorithm 2.20 can simply be modified by not adding a handle for the final triangle, but a twisted handle (two crosscaps) instead [16]. The cycle $uAvwBwC$ is then expanded to $uAvw\overline{B}vu\overline{C}w$, where \overline{B} and \overline{C} denote that B and C are traversed in the opposite direction, respectively.

Example 2.21. The seven upper triangles 124, 137, 156, 235, 267, 346, 457 of Figure 1 form (up to isomorphism) the unique Steiner triple system STS(7) on 7 vertices. Once an embedding for the star of vertex 1, consisting of the triangles 124, 137, 156, is chosen, four handles are added along with the remaining four triangles, and the boundary cycle is expanded each time. Initially, we have 124137156 as boundary cycle, which is expanded as follows:

$$\begin{aligned}
 235 : & \quad 1 \boxed{2} 4 \boxed{1} \boxed{3} 7 \boxed{1} \boxed{5} 6 & \longrightarrow & \quad 1 \boxed{2} \boxed{3} 7 \boxed{1} \boxed{5} \boxed{2} 4 \boxed{1} \boxed{3} \boxed{5} 6, \\
 267 : & \quad 1 \boxed{2} \boxed{3} 7 \boxed{1} 5 2 4 \boxed{1} 3 \boxed{5} 6 & \longrightarrow & \quad 1 \boxed{2} \boxed{7} \boxed{1} 5 2 4 \boxed{1} 3 \boxed{5} \boxed{6} \boxed{2} \boxed{3} \boxed{7} \boxed{6}, \\
 346 : & \quad 1 2 7 \boxed{1} 5 \boxed{2} \boxed{4} \boxed{1} \boxed{3} \boxed{5} \boxed{6} 2 3 7 6 & \longrightarrow & \quad 1 2 7 \boxed{1} 5 \boxed{2} \boxed{4} \boxed{3} \boxed{5} \boxed{6} \boxed{4} \boxed{1} \boxed{3} \boxed{6} 2 3 7 6, \\
 457 : & \quad 1 2 \boxed{7} \boxed{1} \boxed{5} \boxed{2} \boxed{4} \boxed{3} 5 6 4 \boxed{1} 3 6 2 3 7 6 & \longrightarrow & \quad 1 2 \boxed{7} \boxed{5} \boxed{2} \boxed{4} \boxed{7} \boxed{1} \boxed{5} \boxed{4} \boxed{3} 5 6 4 \boxed{1} 3 6 2 3 7 6.
 \end{aligned}$$

The interior of the resulting cycle 127524715435641362376 is triangulated and these ‘white’ triangles together with the ‘black’ triangles of the Steiner triple system give a triangulation of the orientable genus 4 surface that contains the Steiner triple system.

The smallest known Steiner triple system with $\chi_2 = 5$ is PG(64) on 63 vertices, yielding triangulated surfaces (orientable and non-orientable) with $f = (2017, 9765, 6510)$ that have $\chi_2 = 5$.

Theorem 2.22. *The orientable surface of genus $g = 620$ and the non-orientable surface of genus $u = 1240$ have triangulations with $f = (2017, 9765, 6510)$ and $\chi_2 = 5$.*

The smallest known Steiner triple system with $\chi_2 = 6$ is AG(5, 3) on $3^5 = 243$ vertices [9].

Theorem 2.23. *The orientable surface of genus $g = 9680$ and the non-orientable surface of genus $u = 19360$ have triangulations with $f = (29647, 147015, 98010)$ and $\chi_2 = 6$.*

For the orientable cases, list of facets PG64_n2017_o1_g620 and AG_5_3_n29647_o1_g9680 can be found online at [3].

As mentioned above, *maximum genus embeddings* have genus $\frac{1}{6}(n-1)(n-3)$ in the orientable and genus $\frac{1}{3}(n-1)(n-3)$ in the non-orientable case.

Minimum genus embeddings of Steiner triple systems are obtained by complementing a given Steiner triple system STS(n) with a transversal and isomorphic Steiner triple system STS(n)^T on the same number of vertices so that the union STS(n) \cup STS(n)^T triangulates a surface with face vector

$$(2.24) \quad f = \left(n, \binom{n}{2}, \frac{2}{3}\binom{n}{2}\right)$$

and $\chi_E = 2 - \frac{1}{6}(n-3)(n-4)$. These bi-embeddings give orientable genus $\frac{1}{12}(n-3)(n-4)$ surfaces or non-orientable genus $\frac{1}{6}(n-3)(n-4)$ surfaces; cf. [15, 27, 33]. However, transversals need not exist in general.

The following section provides background and results on Steiner triple systems and transversals. In Section 4, we show that the examples in the infinite series of neighborly Bose Steiner surfaces all have $\chi_2 = 3$. In Section 5, we will see that the projective Steiner triple system PG(128) with $\chi_2(\text{PG}(128)) \in \{5, 6\}$ admits a non-orientable transversal PG(128)^T, thus yielding a non-orientable surface PG(128) \cup PG(128)^T of genus 2542 with $f = (127, 8001, 5334)$ and 2-chromatic number $\chi_2(\text{PG}(128) \cup \text{PG}(128)^T) \in \{5, 6\}$.

3. STEINER TRIPLE SYSTEMS AND CHROMATIC NUMBERS OF STEINER SURFACES

This section contains a systematic study of Steiner triple systems and *Steiner surfaces*, in particular, with the aim to clarify some inconsistencies in the literature (see the end of Section 5 for further comments). We say that two Steiner triple systems on a set V are *transversal* (*orientable transversal*) if their union is an (orientable) combinatorial surface with vertex set V . We develop criteria for transversality and orientability and show that transversals to the Steiner triple system μ are indexed by the double coset $\Sigma(\mu) \backslash \Sigma(V) / \Sigma(\mu)$ where $\Sigma(\mu)$ is the automorphism group of μ and $\Sigma(V)$ the automorphism group of the set V . In Section 5 we use this characterization to carry out systematic searches for (orientable) transversals to Steiner triple systems.

Let V be a finite set with an odd number $n = 2m + 1$ of elements. Let STS(V) be the set of all Steiner triple systems on V .

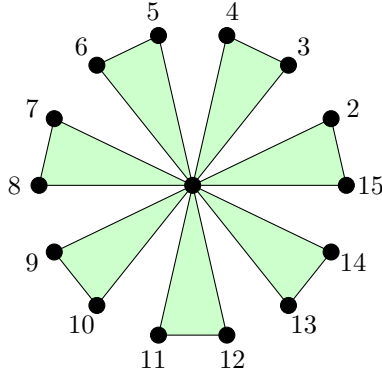
Definition 3.1. *A Steiner quasigroup on V is a binary operation $\mu: V \times V \rightarrow V: (x, y) \rightarrow x \cdot y$ such that*

- $x \cdot x = x$,
- $x \cdot y = y \cdot x$,
- $x \cdot (x \cdot y) = y$

for all $x, y \in V$.

We shall use the following notation:

- $\Sigma(V)$ is the group of all permutations of V .
- $S(\mu) = \{\{x, y, \mu(x, y)\} \mid x, y \in V, x \neq y\}$ is the Steiner triple system associated to the Steiner quasigroup μ . (We shall often not distinguish between a Steiner quasigroup μ on V and the associated Steiner triple system $S(\mu)$ on V .)
- If $\mu \in \text{STS}(V)$ and $T \in \Sigma(V)$ then $\mu^T \in \text{STS}(V)$ is the Steiner quasigroup given by $\mu^T(x, y) = \mu(x^{T^{-1}}, x^{T^{-1}})^T$ for all $x, y \in V$.
- $\Sigma(\mu) = \{a \in \Sigma(V) \mid \forall x, y \in V: \mu(x^a, y^a) = \mu(x, y)^a\}$ is the symmetry group of the Steiner quasigroup μ .

FIGURE 5. A vertex star in a Steiner triple system on $n = 15$ vertices.

There is a right action of the symmetry group of V on the set of Steiner triple systems on V ,

$$(3.2) \quad \text{STS}(V) \times \Sigma(V) \rightarrow \text{STS}(V),$$

taking (μ, T) to μ^T . The isotropy subgroup at μ is $\Sigma(\mu)$: $T \in \Sigma(\mu) \iff \mu^T = \mu$. We have $\mu^T(x^T, y^T) = \mu(x, y)^T$ and $S(\mu)^T = S(\mu^T)$ for all $\mu \in \text{STS}(V)$ and all $T \in \Sigma(V)$.

We note the following well-known result from design theory.

Theorem 3.3. $\text{STS}(V) \neq \emptyset \iff n \equiv 1 \pmod{6} \text{ or } n \equiv 3 \pmod{6}$.

Suppose that S^- and S^+ are two Steiner triple systems on V . We say that S^- and S^+ are disjoint when $S^- \cap S^+ = \emptyset$. Let $S^- \cup S^+$ denote the abstract simplicial complex generated by their union. If S^- and S^+ are disjoint, $S^- \cup S^+$ is a 1-neighborly pseudo-surface (as every 1-simplex lies in exactly two 2-simplices) of Euler characteristic

$$(3.4) \quad \chi(S^- \cup S^+) = n - \binom{n}{2} + \frac{2}{3} \binom{n}{2} = -\frac{n(n-7)}{6}.$$

The Euler characteristic is odd when $n \equiv 1 \pmod{12}$ or $n \equiv 9 \pmod{12}$ and even when $n \equiv 3 \pmod{12}$ or $n \equiv 7 \pmod{12}$.

Definition 3.5. An (orientable) Steiner surface on V is a pair, $S^-, S^+ \in \text{STS}(V)$, of disjoint Steiner triple systems on V so that $S^- \cup S^+$ is an (orientable) combinatorial surface.

An (orientable) transversal to the Steiner triple system $S \in \text{STS}(V)$ is a symmetry $T \in \Sigma(V)$ such that (S, S^T) is an (orientable) Steiner surface on V .

The genus of a Steiner surface on V is $\frac{1}{6}(n-4)(n-3)$ if it is nonorientable and $\frac{1}{12}(n-4)(n-3)$ if it is orientable. Orientable Steiner surfaces exist only for $n \equiv 3, 7 \pmod{12}$.

Suppose that S^- and S^+ are two Steiner triple systems on V and μ^- and μ^+ the corresponding Steiner quasi-groups on V . The transition map

$$(3.6) \quad \sigma: \text{STS}(V) \times V \times \text{STS}(V) \rightarrow \Sigma(V), \quad y^{\sigma(\mu^-, x, \mu^+)} = \mu^-(x, \mu^+(x, y)), \quad x, y \in V, \quad \mu^-, \mu^+ \in \text{STS}(V),$$

records the transition from μ^+ to μ^- in that $\mu^+(x, y)^\sigma = \mu^-(x, y)$ and $\mu^-(x, y^\sigma) = \mu^+(x, y)$ where $\sigma = \sigma(\mu^-, x, \mu^+)$. Note also that $\mu^+(x, y^\sigma)^\sigma = \mu^+(x, y)$ and $\mu^+(x, y^{\sigma^j})^{\sigma^j} = \mu^+(x, y)$ for all natural numbers j by induction.

Proposition 3.7. (μ^-, μ^+) is a Steiner surface on $V \iff \forall x \in V: \sigma(\mu^-, x, \mu^+)$ has cycle structure $1^1 m^2$.

Proof. Assume that $\sigma(\mu^-, x, \mu^+)$ has cycle structure $1^1 m^2$. Suppose that $x, y \in V$, $x \neq y$, and $\mu^-(x, y) = \mu^+(x, y)$. Then $\sigma(\mu^-, x, \mu^+)$ fixes $\mu^-(x, y)$ so $x = \mu^-(x, y)$. This contradiction shows that μ^- and μ^+ define disjoint Steiner triple systems. Now consider the link at x in the simplicial complex $S(\mu^-) \cup S(\mu^+)$. The orbits of $\sigma = \sigma(\mu^-, x, \mu^+)$ through y and $\mu^+(x, y)$ consist of the m distinct points $y, y^\sigma, \dots, y^{\sigma^{m-1}}$ and $\mu^+(x, y), \mu^+(x, y)^\sigma, \dots, \mu^+(x, y)^{\sigma^{m-1}}$, respectively. Observe that the permutations $y \rightarrow \mu^+(x, y)^{\sigma^j}$ fix x and only x as they are conjugate to one of the permutations $y \rightarrow \mu^\pm(x, y)$. Thus the orbits of σ through y and $\mu^+(x, y)$ are disjoint (Figure 6). It follows that the $2m$ vertices in the walk

$$y \rightarrow \mu^+(x, y) \rightarrow y^\sigma \rightarrow \dots \rightarrow \mu^+(x, y^{\sigma^{m-1}}) = \mu^+(x, y)^\sigma \rightarrow y^{\sigma^{m-1}}$$

are all distinct. We conclude that the link at x is the cyclic triangulation of S^1 on $2m$ vertices.

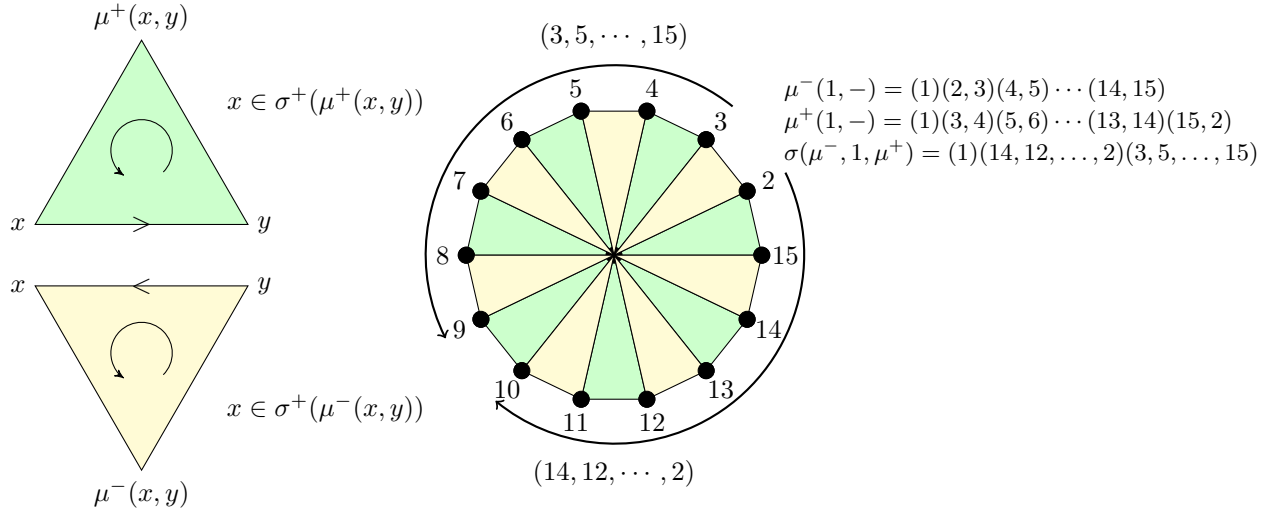


FIGURE 6. Local orientation.

Conversely, assume that (μ^-, μ^+) is a Steiner surface. Let $x, y \in V$, $x \neq y$. Since x belongs to m triples from μ^\pm the link at x is a $2m$ -cycle

$$y \rightarrow \mu^+(x, y) \rightarrow y^\sigma \rightarrow \mu^+(x, y)^\sigma \rightarrow \cdots \rightarrow y^{\sigma^m} = y$$

starting at any point $y \neq x$. Thus the orbit of σ through y has size m . \square

Given an action of a group H on V , a *transversal* is a subset $V_H \subseteq V$ containing exactly one element from each H -orbit in V .

Lemma 3.8. *Suppose that $A \leq \Sigma(\mu^-) \cap \Sigma(\mu^+)$. Then (μ^-, μ^+) is a Steiner surface if and only if*

$$\forall y \in V_A: \sigma(\mu^-, x, \mu^+) \text{ has cycle structure } 1^1 m^2,$$

where $V_A \subseteq V$ is transversal to the action of A on V .

Proof. For any $a \in \Sigma(\mu^-, \mu^+)$, $\sigma(\mu^-, x^a, \mu^+) = \sigma(\mu^-, x, \mu^+)^a$ for all $x \in V$. \square

Definition 3.9. *Suppose (μ^-, μ^+) is a Steiner surface on V .*

- A local orientation for (μ^-, μ^+) at $x \in V$ is a size m orbit $\sigma^+(x)$ for the transition permutation $\sigma(\mu^-, x, \mu^+)$.
- An orientation for (μ^-, μ^+) is a function, $x \rightarrow \sigma^+(x)$, that to every $x \in V$ associates a local orientation $\sigma^+(x)$ at x in such a way that

$$\forall x, y \in V: x \in \sigma^+(\mu^+(x, y)) \iff x \in \sigma^+(\mu^-(x, y)).$$

- An automorphism f of a Steiner surface (μ^-, μ^+) with orientation $x \rightarrow \sigma^+(x)$ is orientation preserving if $f(\sigma^+(x)) = \sigma^+(f(x))$ for all $x \in V$.

Proposition 3.10. *Suppose (μ^-, μ^+) is a Steiner surface on V .*

The combinatorial surface $S(\mu^-) \cup S(\mu^+)$ admits an orientation $\iff (\mu^-, \mu^+)$ admits an orientation.

Proof. Let $V \ni x \rightarrow \sigma^+(x)$ be an orientation of the Steiner surface (μ^-, μ^+) . For every $x \in V$, let $\sigma^-(x)$ denote the size m orbit distinct from $\sigma^+(x)$ of the permutation $\sigma(\mu^-, x, \mu^+)$. Whenever x and y are distinct points of V and $x \in \sigma^\pm(y)$, declare $(x, \mu^\pm(x, y), y)$ to be a positive permutation of the vertices of the 2-simplex $\{x, \mu^\pm(x, y), y\} \in S(\mu^-) \cup S(\mu^+)$. This is a consistent orientation of the 2-simplices of the combinatorial surface $S(\mu^-) \cup S(\mu^+)$.

If the surface $S(\mu^-) \cup S(\mu^+)$ is oriented then the induced orientation of each vertex link picks out one of the m -cycles of that vertex link. \square

Let $a \in \Sigma(\mu^-)$ be a symmetry of the Steiner quasigroup μ^- and let $x, y \in V$. From

$$y^{a\sigma(\mu^-, x^a, \mu^+)} = \mu^-(x^a, \mu^+(x^a, y^a)) = \mu^-(x^a, (\mu^+)^{a^{-1}}(x, y)^a) = \mu^-(x^a, (\mu^+)^{a^{-1}}(x, y))^a = y^{\sigma(\mu^-, x, (\mu^+)^{a^{-1}})^a}$$

we see that $\sigma(\mu^-, x^a, \mu^+) = \sigma(\mu^-, x, {}^a\mu^+)^a$ where we write ${}^a\mu^+ = (\mu^+)^{a^{-1}}$ for the left action of $\Sigma(\mu^-)$ on $\text{STS}(V)$. Thus (μ^-, μ^+) is an (orientable) Steiner surface if and only if $(\mu^-, {}^a\mu^+)$ is an (orientable) Steiner surface. (If $x \rightarrow \sigma^+(x)$ is a local orientation for (μ^-, μ^+) then $x \rightarrow \sigma(x^a)^{a^{-1}}$ is a local orientation for $(\mu^-, {}^a\mu^+)$.)

The following lemma shows that a Steiner surface is orientable if and only if the local orientation at any point propagates coherently to local orientations at all other points.

Lemma 3.11. *Suppose that (μ^-, μ^+) is a Steiner surface on V . Let v be a vertex in V and $\sigma^+(v)$ a local orientation at v . Then (μ^-, μ^+) is orientable if and only if*

$$\forall u_1, u_2 \in \sigma^+(v) \forall x \in v^{\langle \sigma(\mu^-, u_1, \mu^+) \rangle} \cap v^{\langle \sigma(\mu^-, u_2, \mu^+) \rangle} : u_1^{\langle \sigma(\mu^-, x, \mu^+) \rangle} = u_2^{\langle \sigma(\mu^-, x, \mu^+) \rangle}$$

or, equivalently, if and only if the projection onto the first coordinate

$$\bigcup_{u \in \sigma^+(v)} \{(x, u^{\langle \sigma(\mu^-, x, \mu^+) \rangle}) \mid x \in v^{\langle \sigma(\mu^-, u, \mu^+) \rangle}\} \rightarrow V$$

is a bijection.

Proof. The condition of Definition 3.9 for an orientation is that

$$u \in \sigma^+(v) \iff u \in \sigma^+(v^{\sigma(u)})$$

for all distinct vertices $u, v \in V$. (Put $u = x$ and $v = \mu^+(x, y)$ so that $\mu^-(x, y) = \mu^+(x, y)^{\sigma(x)} = v^{\sigma(x)} = v^{\sigma(u)}$.) Repeated application of this gives

$$u \in \sigma^+(v) \iff u \in \sigma^+(v^{\sigma(u)^i})$$

or, equivalently,

$$u \in \sigma^+(v) \iff \forall x \in v^{\langle \sigma(u) \rangle} : u \in \sigma^+(x).$$

This means that $\sigma^+(x)$ is the orbit through u of $\sigma(x)$ for all x in the orbit through v of $\sigma(u)$. \square

Lemma 3.12. *Suppose that $A \leq \Sigma(\mu^-, \mu^+)$ for some $A \leq \Sigma(V)$. Let $s^+ \in \Sigma(V)$ be a fixed-point free permutation of V centralizing A . Then $V \ni x \rightarrow s^+(x)^{\langle \sigma(\mu^-, x, \mu^+) \rangle}$ is an orientation for the Steiner surface (μ^-, μ^+) if and only if*

$$\forall x \in V \forall y \in V_A : x \in s^+(\mu^+(x, y))^{\langle \sigma(\mu^-, x, \mu^+) \rangle} \iff x \in s^+(\mu^-(x, y))^{\langle \sigma(\mu^-, x, \mu^+) \rangle},$$

where $V_A \subseteq V$ is transversal to the action of A on V .

Proof. Put $\sigma(x) = \sigma(\mu^-, x, \mu^+)$ and $\sigma^+(x) = s^+(x)^{\langle \sigma(x) \rangle}$. From the proof of Lemma 3.8, $\sigma(x^a) = \sigma(x)^a$ for all $a \in A$. Using the assumption that $s^+(x^a) = s^+(x)^a$, we see that $\sigma^+(\mu^\pm(x^a, y^a)) = \sigma^+(\mu^\pm(x, y))^a$ for all $x, y \in V$ and $a \in A$. Let $V^\pm(y) = \{x \in V \mid x \in \sigma^+(\mu^\pm(x, y))\}$. We have shown that $V^\pm(y^a) = V^\pm(y)^a$, and thus $V^+(y) = V^-(y) \iff V^+(y^a) = V^-(y^a)$ for all $y \in V$. The lemma follows since $V \ni x \rightarrow \sigma^+(x)$ is an orientation for (μ^-, μ^+) if and only if $V^+(y) = V^-(y)$ for all $y \in V$. \square

Let μ be a Steiner quasigroup on V . We shall now discuss (orientable) transversals to μ (Definition 3.5).

The shift map of μ ,

$$(3.13) \quad \sigma : V \times \Sigma(V) \rightarrow \Sigma(V), \quad y^{\sigma(x, T)} = \mu(x, \mu^T(x, y)), \quad x, y \in V, \quad T \in \Sigma(V),$$

takes (x, T) to the transition permutation $\sigma(\mu, x, \mu^T)$ from μ^T to μ . Thus $\mu^T(x, y)^{\sigma(x, T)} = \mu(x, y)$. We call $\sigma(x, T)$ the x -shift of T and σ .

Proposition 3.14. *Let μ be a Steiner quasigroup on V and $T \in \Sigma(V)$ a symmetry of V .*

$$T \text{ is transversal to } \mu \iff \forall x \in V : \sigma(x, T) \text{ has cycle structure } 1^1 m^2.$$

If T is transversal to μ then T is orientably transversal to μ if and only if there are transitive size m orbits $\sigma^+(x, T)$ for $\sigma(x, T)$ at every $x \in V$ such that

$$x^T \in \sigma^+(\mu(x^T, y^T), T) \iff x^T \in \sigma^+(\mu(x, y)^T, T)$$

for all $x, y \in V$.

The automorphism group of μ^T is $\Sigma(\mu^T) = \Sigma(\mu)^T$. The intersection

$$(3.15) \quad \Sigma(\mu) \cap \Sigma(\mu)^T = \{a \in \Sigma(V) \mid \mu^a = \mu, \mu^{Ta} = \mu^T\}$$

acts on the simplicial complex $S(\mu) \cup S(\mu)^T$. If T is transversal to μ then $\Sigma(\mu) \cap \Sigma(\mu)^T$ is contained in the automorphism group of the combinatorial surface $S(\mu) \cup S(\mu)^T$ with index at most 2.

Proposition 3.16. *Let $T \in \Sigma(V)$ be a symmetry of V .*

(1) *If $a \in \Sigma(V)$ is a symmetry of V then*

$$a \in \Sigma(\mu) \iff \mu^a = \mu \iff \mu^{aT} = \mu^T \iff \forall x \in V: \sigma(x, aT) = \sigma(x, T) \iff \forall x \in V: \sigma(x^a, T^a) = \sigma(x, T)^a.$$

(2) *If $a \in \Sigma(\mu)$ is a symmetry of μ then*

$$a \in \Sigma(\mu) \cap \Sigma(\mu)^T \iff \mu^{Ta} = \mu^T \iff \forall x \in V: \sigma(x, Ta) = \sigma(x, T) \iff \forall x \in V: \sigma(x^a, T) = \sigma(x, T)^a.$$

Proof. (1) The first part is clear since $\Sigma(\mu)$ is the isotropy subgroup at μ under the right action (3.2) of $\Sigma(V)$ on $\text{STS}(V)$. The condition $\sigma(x^a, T^a) = \sigma(x, T)^a$ for all $x \in V$ is equivalent to $\mu^{T^a}(x^a, y^a) = \mu^T(x, y)^a$ or $\mu^a(x^{T^{-1}}, y^{T^{-1}}) = \mu(x^{T^{-1}}, y^{T^{-1}})$ for all $x, y \in V$.

(2) Clearly, $a \in \Sigma(\mu^T) \iff \mu^{Ta} = \mu^T$, as $\Sigma(\mu^T)$ is the isotropy subgroup at μ^T . Next

$$\mu^{Ta} = \mu^T \iff \forall x, y \in V: \mu(x, \mu^{Ta}(x, y)) = \mu(x, \mu^T(x, y)) \iff \forall x \in V: \sigma(x, T) = \sigma(x, Ta)$$

establishes the second equivalence. Since $a \in \Sigma(\mu)$ we have, as in (1), that $\mu^T(x, y)^a = \mu^{Ta}(x^a, y^a)$ for all $x, y \in V$. This leads to a new chain of equivalences,

$$\begin{aligned} \forall x \in V: \sigma(x^a, T) = \sigma(x, T)^a &\iff \forall x, y \in V: \mu(x^a, \mu^T(x^a, y^a)) = \mu(x, \mu^T(x, y))^a \iff \\ &\iff \forall x, y \in V: \mu(x^a, \mu^T(x^a, y^a)) = \mu(x^a, \mu^{Ta}(x^a, y^a)) \iff \mu^{Ta} = \mu^T, \end{aligned}$$

proving the final equivalence. \square

Remark 3.17. Suppose that $a \in \Sigma(\mu)$, $T \in \Sigma(V)$ is transversal to μ , and that $\sigma(x_0^a, T) = \sigma(x_0, T)^a$ for some $x_0 \in V$. Let $y_0 \neq x_0$ be a point of V distinct from x_0 . For any $j \geq 0$,

$$y_0^{\sigma(x_0, T)^j} \xrightarrow{a} Y_0^{\sigma(X_0, T)^j}, \quad \mu(x_0, y_0)^{\sigma(x_0, T)^j} \xrightarrow{a} \mu(X_0, Y_0)^{\sigma(X_0, T)^j}.$$

Thus a is completely determined by its values, $X_0 = x_0^a$ and $Y_0 = y_0^a$, on the two points x_0 and y_0 . (The points y_0 and $\mu(x_0, y_0)$ are not in the same orbit of $\sigma(x_0, T)$.) Therefore the order of $\Sigma(\mu) \cap \Sigma(\mu)^T$ is at most $n(n-1)$.

Let $a \in \Sigma(\mu)$ be a symmetry of μ . Proposition 3.16.(1) shows that

$$\begin{aligned} T \text{ is (orientably) transversal to } \mu &\iff aT \text{ is (orientably) transversal to } \mu \iff \\ &\iff T^a \text{ is (orientably) transversal to } \mu \iff Ta \text{ is (orientably) transversal to } \mu. \end{aligned}$$

Note that if $\sigma^+(x)$ is an orientation for (μ, μ^T) then $\sigma^+(x^{a^{-1}})^a$ is an orientation of (μ, μ^{Ta}) . Thus transversals to the Steiner quasigroup μ are really elements of $\Sigma(\mu) \backslash \Sigma(V) / \Sigma(\mu)$.

Definition 3.18. *For any Steiner quasigroup μ on V ,*

$$T^{(+)}(\mu) = \{\Sigma(\mu)T\Sigma(\mu) \in \Sigma(\mu) \backslash \Sigma(V) / \Sigma(\mu) \mid T \text{ is (orientably) transversal to } \mu\}$$

denotes the set of equivalence classes of (orientable) transversals to μ . For any subgroup $A \leq \Sigma(\mu)$,

$$T^{(+)}(\mu)_{\gtrsim A} = \{T \in T^{(+)}(\mu) \mid \Sigma(\mu) \cap \Sigma(\mu)^T \gtrsim A\}$$

is the set of (orientable) transversals T for which $\Sigma(\mu) \cap \Sigma(\mu)^T$ is superconjugate in $\Sigma(\mu)$ to A .

The next Proposition 3.19 deals with sets of double cosets of the form $H \backslash G / H$. The following notation will be used: When G is a group and $A \leq H \leq G$ and $K \leq G$ are subgroups of G then

- $K \gtrsim_G H$ means that K is superconjugate in G to H (a G -conjugate of K is a supergroup of H),
- $N_G(H, K) = \{g \in G \mid H^g \leq K\}$ is the transporter set,
- $A^H = \{A^h \mid h \in H\}$ is the set of H -conjugates of A ,
- $A_{\leq H}^G$ is the set of G -conjugates of A contained in H .

Proposition 3.19. *Let H be a subgroup of G and A a subgroup of H .*

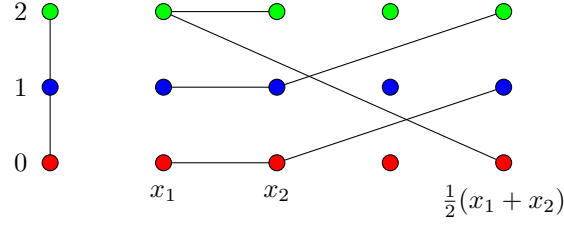
(1) *The map*

$$N_G(A, H) \xrightarrow{g \mapsto Hg^{-1}H} \{g \in H \backslash G / H \mid H \cap H^g \gtrsim A\}$$

is surjective.

(2) *Let $g_1, \dots, g_t \in N_G(A) \backslash G / H$ be distinct elements such that $A_{\leq H}^G = \{A^{g_1}, \dots, A^{g_t}\}$. Then*

$$N_G(A, H) = \coprod_{1 \leq j \leq t} g_j N_G(A^{g_j}) H.$$

FIGURE 7. The Bose Steiner triple systems $B(s)$ are $(3, 2)$ -colorable.

(3) If $A_{\leq H}^G \subseteq A^H$ (i.e., every G -conjugate of A in H is H -conjugate to A) then

$$N_H(A) \backslash N_G(A) / N_H(A) \xrightarrow{N_H(A)gN_H(A) \rightarrow HgH} \{g \in H \backslash G / H \mid H \cap H^g \gtrsim A\}$$

is surjective and it is bijective if also $N_G(A) \leq N_G(H)$.

Proof. (1) If $g \in N_G(A, H)$ then $A^g \leq H$ or $H^{g^{-1}} \geq H$. Thus the map $g \rightarrow g^{-1}$ takes the set on the left to the set on the right. This map is surjective because

$$H^g \gtrsim A \iff \exists h \in H: H^g \geq A^h \iff \exists h \in H: A^{hg^{-1}} \leq H \iff \exists h \in H: hg^{-1} \in N_G(A, H)$$

and the image of hg^{-1} in $H \backslash G / H$ is HgH .

(2) Let $g \in N_G(A, H)$. Since $A^g \leq H$ is conjugate in H to A^{g_j} for some j , $A^{g_j} = A^{g^h}$ for some $h \in H$. Then ghg_j^{-1} normalizes A , $gh \in N_G(A)g_j$ and $g = (gh)h^{-1} \in N_G(A)g_jH = g_jN_G(A^{g_j})H$.

(3) Since $A_{\leq H}^G = A^H$, $N_G(A, H) = N_G(A)H$ according to (2) and the map

$$N_G(A) \xrightarrow{g \rightarrow HgH} \{g \in H \backslash G / H \mid H \cap H^g \gtrsim A\}$$

is surjective by (1). Suppose that $g_1, g_2 \in N_G(A)$ have identical images in $H \backslash G / H$. Then $h_1g_1 = g_2h_2$ for some $h_1, h_2 \in H$. Thus $g_2^{-1}g_1 = g_2^{-1}h_1^{-1}g_2h_2 = (h_1^{-1})^{g_2}h_2 \in H$ since $H^{g_2} = H$ as all elements of G normalizing A normalize H by assumption. Thus $g_2^{-1}g_1 \in N_H(A)$ and $g_1N_H(A) = g_2N_H(A)$. \square

We conclude from Proposition 3.19.(3) that when $A \leq \Sigma(\mu)$ has the property that any $\Sigma(V)$ -conjugate of A in $\Sigma(\mu)$ is $\Sigma(\mu)$ -conjugate to A (i.e., if A is a Sylow p -subgroup of $\Sigma(\mu)$ for some prime p) then there is a surjection

$$(3.20) \quad N_{\Sigma(\mu)}(A) \backslash \{T \in N_{\Sigma(V)}(A) \mid T \text{ is transversal to } \mu\} / N_{\Sigma(\mu)}(A) \twoheadrightarrow T(\mu)_{\gtrsim A}$$

that is bijective if also $N_{\Sigma(V)}(A) \leq N_{\Sigma(V)}(\Sigma(\mu))$.

Let $S \in \text{STS}(V)$ be a Steiner triple system on V and $T \in \Sigma(V)$ a transversal to S . The 1-chromatic number of the Steiner surface $S \cup S^T$ is $\chi_1(S \cup S^T) = |V| = n$ as all Steiner surface are neighborly. But what about the 2-chromatic number? We discuss this question for two families of Steiner triple systems, the Bose Steiner triple systems and the projective Steiner triple systems.

4. THE BOSE STEINER TRIPLE SYSTEMS — AN INFINITE SERIES OF NEIGHORLY SURFACES WITH $\chi_2 = 3$

We show in this section that Bose Steiner triple systems generate an infinite sequence of neighborly triangulated surfaces with unbounded 1-chromatic numbers, but with constant 2-chromatic number $\chi_2 = 3$.

Let $s > 1$. In the ring $\mathbf{Z}/(2s+1)\mathbf{Z}$ of integers modulo the odd integer $2s+1$, 2 is invertible with $2^{-1} = s+1$. Let $V = \mathbf{Z}/(2s+1)\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}$. Then V has order $n = |V| = 3(2s+1)$ and $n \equiv 3 \pmod{6}$. Let also $m = 3s+1$ so that $n = 2m+1$. The Bose Steiner triple system on V [8],

$$B(s) = \{\{x\} \times \mathbf{Z}/3\mathbf{Z} \mid x \in \mathbf{Z}/(2s+1)\mathbf{Z}\} \\ \cup \{(x_1, y), (x_2, y), ((x_1 + x_2)/2, y+1)\} \mid x_1, x_2 \in \mathbf{Z}/(2s+1)\mathbf{Z}, y \in \mathbf{Z}/3\mathbf{Z}, x_1 \neq x_2\},$$

consists of $2s+1$ ‘vertical’ and $3\binom{2s+1}{2} = 3s(2s+1)$ ‘slanted’ triangles. Because

$$n \pmod{12} = \begin{cases} 3 \pmod{12}, & s \text{ even}, \\ 9 \pmod{12}, & s \text{ odd} \end{cases}$$

Bose systems $B(s)$ with odd s admit nonorientable transversals only, but for even s orientable transversals may exist.

The following proposition and theorem, nearly all of which was proved in [15], contain an example of an *orientable* transversal to $B(s)$. The proof presented here does not use results from topological graph theory so may represent a partial response to [15, Problem 1].

Proposition 4.1. [15, Theorem 1]. *Assume that $s \geq 2$ is even. The permutation*

$$(x, y)^T = \begin{cases} (x, 0), & y = 0, \\ (x + 1, 2), & y = 1, \\ (x + 1, 1), & y = 2 \end{cases}$$

is orientably transversal to $B(s)$.

Proof. Here, we give an alternative proof to [15]. By Proposition 3.14 we must determine the cycle structure of the shifts $\sigma(u, T)$ defined by equation (3.13) for all $u \in V$. However, $\Sigma(\mu) \cap \Sigma(\mu)^T$ contains the subgroup $\mathbf{Z}/(2s+1)\mathbf{Z}$ so by Lemma 3.8 it suffices to do this for the three u in $V_{\mathbf{Z}/(2s+1)\mathbf{Z}} = \{0\} \times \mathbf{Z}/3\mathbf{Z}$.

The formulas

$$\mu((0, 0), (x, y)) = \begin{cases} (\frac{1}{2}x, 1), & y = 0, x \neq 0, \\ (2x, 0), & y = 1, x \neq 0, \\ (-x, 2), & y = 2, x \neq 0, \end{cases} \quad \mu^T((0, 0), (x, y)) = \begin{cases} (\frac{1}{2}x - 1, 2), & y = 0, x \neq 0, \\ (-x + 2, 1), & y = 1, x \neq 1, \\ (2x - 2, 0), & y = 2, x \neq 1 \end{cases}$$

imply that

$$(x, y)^{\sigma((0, 0), T)} = \mu((0, 0), \mu^T((0, 0), (x, y))) = \begin{cases} (-\frac{1}{2}x - 1, 2), & y = 0, x \neq 0, x \neq -2, \\ (-2x + 4, 0), & y = 1, x \neq 1, x \neq 2, \\ (x - 1, 1), & y = 2, x \neq 1. \end{cases}$$

This expression can be used to show that the shift $\sigma((0, 0), T)$ has cycle structure $1^1 m^2$. The orbits through $(0, 1)$ and $\mu((0, 0), (0, 1)) = (0, 2)$,

$$\begin{array}{c} \overbrace{(0, 1) \rightarrow \cdots \rightarrow (1, 1)}^{3s/2} \rightarrow \overbrace{(2s, 2) \rightarrow \cdots \rightarrow (4, 2)}^{3s/2-3} \rightarrow (3, 1) \rightarrow (-2, 0) \rightarrow (0, 1), \\ \overbrace{(0, 2) \rightarrow \cdots \rightarrow (1, 2)}^{3s/2} \rightarrow \overbrace{(2, 0) \rightarrow \cdots \rightarrow (5, 0)}^{3s/2-3} \rightarrow (3, 2) \rightarrow (2, 1) \rightarrow (0, 2), \end{array}$$

are disjoint and both have length m . It may be helpful to observe that $T^3(x, 0) = (x + 8, 0)$, $T^3(x, 1) = (x - 4, 1)$, $T^3(x, 2) = (x - 4, 2)$ provided T^3 does not involve any of the exceptions $(0, 0), (-2, 0), (1, 1), (2, 1), (1, 2)$.

Completely analogous arguments show that also $\sigma((0, 1), T)$ and $\sigma((0, 2), T)$ have cycle structure $1^1 m^2$. We have now shown that T is a transversal to $B(s)$. \square

Theorem 4.2. *The orientable combinatorial Steiner surface $B(s) \cup B(s)^T$ has genus $\frac{1}{2}s(6s - 1)$ and chromatic numbers $\chi_1(B(s) \cup B(s)^T) = 3(2s + 1)$, $\chi_2(B(s) \cup B(s)^T) = 3$. The horizontal shift $h(x, y) = (x + 1, y)$ is an orientation preserving automorphism of $B(s) \cup B(s)^T$.*

Proof. If we let $v(u) = u + (0, 1)$ denote the vertical and $h(u) = u + (1, 0)$ the horizontal shift of V , then $V \ni u \rightarrow v(u)^{\langle \sigma(u, T) \rangle}$ is an orientation for $B(s) \cup B(s)^T$ (Lemma 3.12) and $h \in \Sigma(B(s)) \cap \Sigma(B(s))^T$ is an orientation preserving automorphism of $B(s) \cup B(s)^T$.

The projection $\mathbf{Z}/(2s+1)\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z} \rightarrow \mathbf{Z}/3\mathbf{Z}$ is a $(3, 2)$ -coloring of $B(s) \cup B(s)^T$ which has no horizontal triangles (Figure 7). On the other hand, all Steiner triple system with more than one triple have 2-chromatic number at least 3 [32]. \square

Remark 4.3. Figure 7 illustrates the case $s = 2$. Modulo $2s + 1$ we have $(1 + 2)/2 \equiv 4$. The first coordinates of triangles in the Bose system are either constant, as in the vertical triangle on the left, or, if they are non-vertical, they have first coordinates $(x_1, x_2, (x_1 + x_2)/2)$. The figure shows the three nonvertical triangles with first coordinates $x_1 = 1, x_2 = 2$, and hence the third first coordinate is 4 (never 3).

For the sake of completeness, we quote the following result from [33] exhibiting a *nonorientable* transversal to $B(s)$ for any $s \geq 1$.

Proposition 4.4. [33, Theorem 3] *Assume that $s \geq 1$. The permutation*

$$(x, y)^T = \begin{cases} (x, 0), & y = 0, \\ (x + 1, 1), & y = 1, \\ (x + 2, 2), & y = 2 \end{cases}$$

is nonorientably transversal to $B(s)$.

One may use Lemma 3.11 to show that $B(s) \cup B(s)^T$ is a nonorientable Steiner surface. The statements about its chromatic numbers are proved in the same way as in the proof of Theorem 4.2.

Theorem 4.5. *The nonorientable combinatorial Steiner surface $B(s) \cup B(s)^T$ has genus $s(6s - 1)$ and chromatic numbers $\chi_1(B(s) \cup B(s)^T) = 3(2s + 1)$, $\chi_2(B(s) \cup B(s)^T) = 3$.*

The automorphism group of $B(s)$ contains $(\mathbf{Z}/(2s+1)\mathbf{Z} \times \mathbf{Z}/3\mathbf{Z}) \rtimes (\mathbf{Z}/(2s+1)\mathbf{Z})^\times$ and, in fact, equals this group except for $s = 1$ or $s = 4$ [24, Theorem 3.1].

5. THE PROJECTIVE STEINER TRIPLE SYSTEMS — A NON-ORIENTABLE SURFACE WITH $\chi_2 \in \{5, 6\}$

Let $d > 2$, $N = 2^d$ and $m = 2^{d-1} - 1$ so that $N - 1 = 2m + 1$. Let \mathbf{F}_N be the field with N elements and \mathbf{F}_N^\times the set of nonzero elements in \mathbf{F}_N . The *projective geometry Steiner triple system* is the set

$$\text{PG}(N) = \{\{a, b, c\} \mid a, b, c \in \mathbf{F}_N^\times, \quad a \neq b, \quad a + b + c = 0\}$$

of 3-subsets of the $(N-1)$ -set \mathbf{F}_N^\times . The corresponding Steiner quasigroup is $\mu(a, b) = a + b$, $a \neq b$. The automorphism group of $\text{PG}(N)$ is the linear group $\Sigma(\text{PG}(N)) = \text{GL}(d, \mathbf{F}_2)$. The Singer cycle (multiplication by a primitive element of \mathbf{F}_N) is an automorphism of $\text{PG}(N)$ acting transitively on the vertex set. Because $2^4 \equiv 2^2 \pmod{12}$ and $2^5 \equiv 2^3 \pmod{12}$,

$$2^d - 1 \pmod{12} = \begin{cases} 7 \pmod{12}, & d \text{ odd}, \\ 3 \pmod{12}, & d \text{ even}, \end{cases}$$

so that all projective Steiner triple system potentially allow orientable transversals.

Let $\Sigma_0(\mathbf{F}_N)$ be the group of all permutations of \mathbf{F}_N fixing 0. Including also 0 in the domain, the shift-map (3.13) becomes

$$\sigma: \mathbf{F}_N \times \Sigma_0(\mathbf{F}_N) \rightarrow \Sigma_0(\mathbf{F}_N)$$

given by $y^{\sigma(x, T)} = (y^{-T} + x^{-T})^T + x$ for all $x, y \in \mathbf{F}_N$ and $T \in \Sigma_0(\mathbf{F}_N)$. These shifts have the following properties

- $\sigma(x, \text{Id}) = \text{Id}$ for all $x \in \mathbf{F}_N$,
- $\sigma(0, T) = \text{Id}$ for all $T \in \Sigma_0(\mathbf{F}_N)$,
- $\sigma(x, aT) = \sigma(x, T)$ and $\sigma(x^a, T^a) = \sigma(x, T)^a$ for all $a \in \text{GL}(d, \mathbf{F}_2)$ (Proposition 3.16.(1)),
- $\sigma(x, T)$ fixes 0 and x for all $T \in \Sigma_0(\mathbf{F}_N)$,

where Id is the identity permutation.

Fix an element $z \in \mathbf{F}_N$ distinct from 0 and 1. A permutation $T \in \Sigma_0(\mathbf{F}_N)$ is by Proposition 3.14 transversal to $\text{PG}(N)$ if and only if:

- (1) For all $x \in \mathbf{F}_N^\times$, the x -shift $\sigma(x, T)$ partitions $\mathbf{F}_N - \{0, x\}$ into two equally sized orbits.

It is orientably transversal to $\text{PG}(N)$ if also

- (2) $\forall u_1, u_2 \in z^{\langle \sigma(1, T) \rangle} \forall x \in 1^{\langle \sigma(u_1, T) \rangle} \cap 1^{\langle \sigma(u_2, T) \rangle} : u_1^{\langle \sigma(x, T) \rangle} = u_2^{\langle \sigma(x, T) \rangle}$ according to Lemma 3.11.

The next lemma shows that transversals to $\text{PG}(N)$ are permutations of \mathbf{F}_N that are highly non-linear.

Lemma 5.1. *Let $T \in \Sigma_0(\mathbf{F}_N)$ be a transversal to $\text{PG}(n)$.*

- (1) T is not linear.
- (2) The image, U^T , of any linear subspace U of $\mathbf{F}_N^+ = \mathbf{F}_2^d$ is not a linear subspace when $1 < \dim_{\mathbf{F}_2} U < d$.

Proof. Proposition 3.16 shows that no transversal to $\text{PG}(N)$ can be linear since the identity is not transversal.

Let U be a proper linear subspace of dimension > 1 of the d -dimensional \mathbf{F}_2 -vector space \mathbf{F}_N^+ underlying the field \mathbf{F}_N . Suppose that the image, U^T , of U is again a linear subspace. Choose two distinct and nonzero elements $x, y \in U^T$. This is possible as $\dim_{\mathbf{F}_2} U^T > 1$. Note that the orbit of the x -shift $\sigma(x, T)$ through y stays inside U^T and does not contain 0 and x . The size of the orbit $y^{\sigma(x, T)}$ is at most $|U| - 2 \leq 2^{d-1} - 2 < 2^{d-1} - 1 = m$. Thus $\sigma(x, T)$ cannot have cycle structure $1^2 m^2$. \square

It is convenient to represent permutations of \mathbf{F}_N by polynomials with coefficients in \mathbf{F}_N .

Remark 5.2 (Permutation polynomials and (orientable) surface polynomials). For any permutation $T \in \Sigma_0(\mathbf{F}_N)$ there is a unique permutation polynomial [23] $P_T \in \mathbf{F}_N[X]$ of degree $N - 2$ such that $P_T(x) = x^T$ for all $x \in \mathbf{F}_N$. The coefficients of $P_T = \sum_{i=1}^{N-2} a_i X^i$ are given by

$$(a_1, \dots, a_{N-2})(x^{ij})_{1 \leq i, j \leq N-2} = ((x^j)^T)_{1 \leq j \leq N-2},$$

where x is primitive in \mathbf{F}_N . If $T \in \Sigma_0(\mathbf{F}_N)$ is an (orientable) transversal to $\text{PG}(N)$ we say that P_T is an (orientable) surface polynomial. With this terminology, for a polynomial $P \in \mathbf{F}_N[X]$ we have that

P is an (orientable) surface polynomial $\iff (\text{PG}(N), \text{PG}(N)^P)$ is an (orientable) Steiner surface,

where the Steiner surface has genus $\frac{1}{12}(N-5)(N-4)$ if orientable and the double of this number if nonorientable.

Example 5.3 (Surface monomials). Suppose that $\text{GCD}(N-1, r) = 1$ so that $T = X^r$ is a permutation monomial with $0^T = 0$ and $1^T = 1$. Since $\mu^T(x, y) = (x^s + y^s)^r$, where $rs \equiv 1 \pmod{N-1}$ by Lemma 3.8, we see that $\mathbf{F}_N^\times \leq \Sigma(\mu) \cap \Sigma(\mu)^T$ and hence by Lemma 3.8:

X^r is a surface monomial. \iff The permutation $\sigma(1, X^r): y \rightarrow 1 + (1 + y^s)^r$ of \mathbf{F}_N has cycle structure $1^2 m^2$.

Lemma 5.1 shows that X^r is *not* a surface monomial if

- $d = d_1 d_2$ with $d_1, d_2 > 1$ so that X^r normalizes the subfield $\mathbf{F}_{2^{d_1}}$ of dimension $1 < 2^{d_1} < 2^d$, or,
- r is a power of 2, so that X^r is linear.

Thus surface monomials only occur when d is prime. See [27, Theorem 5] for a list of surface monomials for small values of N . (Surface binomials are much more mysterious.)

Proposition 5.4 ($d = 3$). $|\text{T}(\text{PG}(8))| = 1 = |\text{T}^+(\text{PG}(8))|$. If T is the unique transversal to $\text{PG}(8)$ then $\text{GL}(3, \mathbf{F}_2) \cap \text{GL}(3, \mathbf{F}_2)^T = N_{\text{GL}(3, \mathbf{F}_2)}(\mathbf{F}_{2^3}^\times)$ is the normalizer of the Sylow 7-subgroup of $\text{GL}(3, \mathbf{F}_2)$.

Proof. The quotient set $\text{GL}(3, \mathbf{F}_2) \backslash \Sigma_0(\mathbf{F}_{2^3}) / \text{GL}(3, \mathbf{F}_2)$ has four elements. Using Proposition 3.14 and Lemma 3.11 we see that only one of the four double cosets is transversal to $\text{PG}(8)$ and that this transversal is orientable. \square

$P_3 = X^3$ is one of the orientable transversal to $\text{PG}(8)$. Let a be a primitive element of \mathbf{F}_8 . Then $s^+(x) = ax$ defines an orientation (Lemma 3.12) of the Steiner surface $\text{PG}(8) \cup \text{PG}(8)^{P_3}$ (Figure 8) and $N_{\text{GL}(2, \mathbf{F}_3)}(\mathbf{F}_{2^3}^\times) = \mathbf{F}_{2^3}^\times \rtimes \text{Gal}(\mathbf{F}_{2^3}, \mathbf{F}_2)$ acts by orientation preserving automorphisms.

Proposition 5.5 ($d = 4$). $|\text{T}(\text{PG}(16))| = 4$ and $|\text{T}^+(\text{PG}(16))| = 1$. If T is a nonorientable transversal then $\text{GL}(4, \mathbf{F}_2) \cap \text{GL}(4, \mathbf{F}_2)^T$ is trivial in two cases and is the Sylow 5-subgroup of $\text{GL}(4, \mathbf{F}_2)$ in one case. If T is an orientable transversal then $\text{GL}(4, \mathbf{F}_2) \cap \text{GL}(4, \mathbf{F}_2)^T$ is the Sylow 5-subgroup of $\text{GL}(4, \mathbf{F}_2)$.

Proof. It is possible with a computer to go through all $T \in \text{GL}(4, \mathbf{F}_2) \backslash \Sigma_0(\mathbf{F}_{2^4}) / \text{GL}(4, \mathbf{F}_2)$ and use Proposition 3.14 to check if T is an orientable transversal. This gives an explicit description of the set $\text{T}(\text{PG}(16))$ of transversals. \square

$P_4 = aX^{11} + X^6 + X$, where a is any primitive elements of \mathbf{F}_{16} such that $a^4 + a + 1 = 0$, is an orientable transversal to $\text{PG}(16)$ and $\text{GL}(4, \mathbf{F}_2) \cap \text{GL}(4, \mathbf{F}_2)^{P_4} = \langle a^3 \rangle$ is the Sylow 5-subgroup of $\mathbf{F}_{2^4}^\times$ and $\text{GL}(4, \mathbf{F}_2)$. The primitive element a can be chosen such that $s^+(x) = a^5 x$ defines an orientation (Lemma 3.12) on the orientable Steiner surface $\text{PG}(16) \cup \text{PG}(16)^{P_4}$ such that $\text{GL}(4, \mathbf{F}_2) \cap \text{GL}(4, \mathbf{F}_2)^{P_4}$ acts by orientation preserving automorphisms. There are no surface monomials as d is a composite number (Example 5.3).

Proposition 5.6 ($d = 5$, [27, Example 1]). Let p is a prime divisor of $|\text{GL}(5, \mathbf{F}_2)|$ and S_p the Sylow p -subgroup of $\text{GL}(5, \mathbf{F}_2)$. Then $|\text{T}^+(\text{PG}(32))_{\geq S_p}| = 0$ and

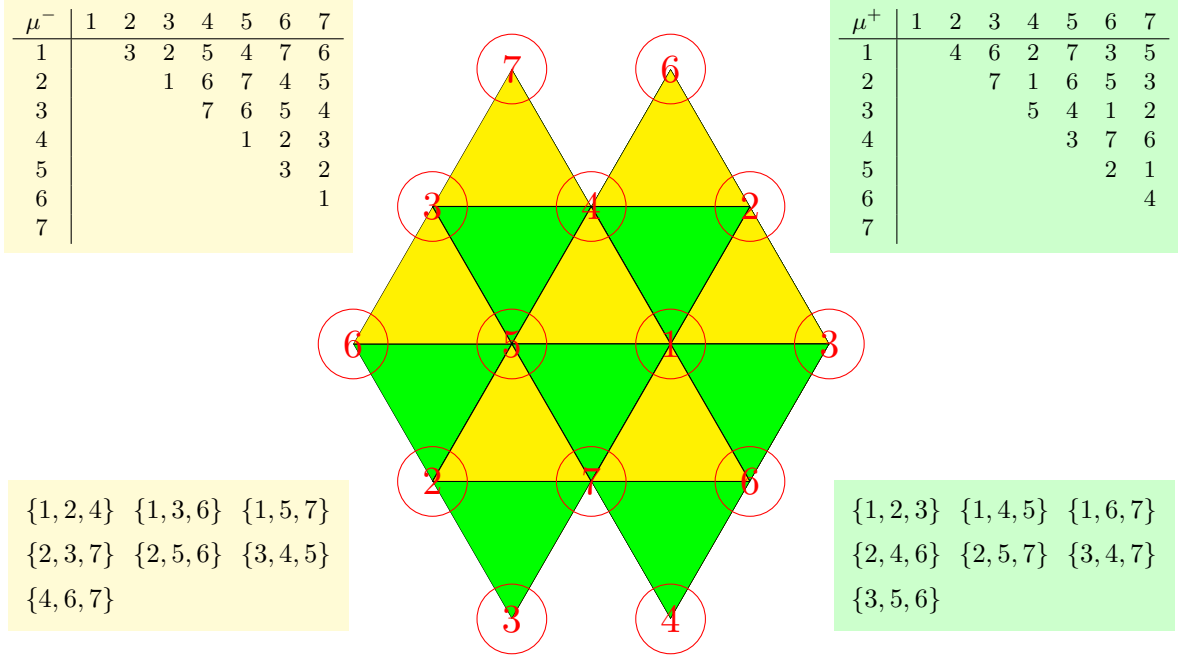
$$|\text{T}(\text{PG}(32))_{\geq S_p}| = \begin{cases} 2, & p = 31, \\ 65, & p = 5, \\ 0, & p = 2, 3, 7. \end{cases}$$

For any $T \in \text{T}(\text{PG}(32))_{\geq S_{31}}$, $\text{GL}(5, \mathbf{F}_2) \cap \text{GL}(5, \mathbf{F}_2)^T = N_{\text{GL}(5, \mathbf{F}_2)}(\mathbf{F}_{32}^\times) = \mathbf{F}_{32}^\times \rtimes \text{Gal}(\mathbf{F}_{2^5}, \mathbf{F}_2)$. For any $T \in \text{T}(\text{PG}(32))_{\geq S_5}$, $\text{GL}(5, \mathbf{F}_2) \cap \text{GL}(5, \mathbf{F}_2)^T$ is $\text{Gal}(\mathbf{F}_{2^5}, \mathbf{F}_2)$ (in 63 cases) or $N_{\text{GL}(5, \mathbf{F}_2)}(\mathbf{F}_{32}^\times)$ (in 2 cases).

Proof. When $d = 5$ it is no longer feasible to search for transversals in the complete set $\text{GL}(d, \mathbf{F}_2) \backslash \Sigma_0(\mathbf{F}_{2^d}) / \text{GL}(d, \mathbf{F}_2)$ of double cosets as we did for $d = 4$. However, by (3.20) all transversals in $\text{T}(\mu)_{\geq S_p}$ are represented by double cosets of the smaller set $N_{\text{GL}(5, \mathbf{F}_2)}(S_p) \backslash N_{\Sigma_0(\mathbf{F}_{2^5})}(S_p) / N_{\text{GL}(5, \mathbf{F}_2)}(S_p)$, and it is possible with a computer to locate the coset representatives of (orientable) transversals using the tests of Proposition 3.14. We find that the set $\text{T}(\text{PG}(32))_{\geq S_p}$ is empty except for $p = 31$ and $p = 5$. \square

$P_5 = X^5$ is a nonorientable transversal to $\text{PG}(32)$. No orientable transversals to $\text{PG}(32)$ are known.

Proposition 5.7 ($d = 7$, [27, Example 2]). $|\text{T}(\text{PG}(128))_{\geq \mathbf{F}_{2^8}^\times}| = 8$. For any $T \in \text{T}(\text{PG}(128))_{\geq \mathbf{F}_{2^8}^\times}$, T is nonorientable and the group $\text{GL}(7, \mathbf{F}_2) \cap \text{GL}(7, \mathbf{F}_2)^T = N_{\text{GL}(7, \mathbf{F}_2)}(\mathbf{F}_{2^7}^\times)$. One of these transversals has permutation monomial $P_7 = X^7$.

FIGURE 8. The orientable Steiner surface $(PG(8), PG(8)^{P_3})$ of genus 1

Proof. We proceed as in the proof of Proposition 5.6. □

When $d = 6, 8, 9$, **no** surface polynomials for \mathbf{F}_{2^d} are known and surface monomials do not exist (Lemma 5.1). Here is what we know about the 2-chromatic numbers:

- For all $d \geq 4$, $3 \leq \chi_2(PG(2^d)) \leq \chi_2(PG(2^{d+1})) \leq \chi_2(PG(2^d)) + 1$ and $\chi_2(PG(2^d)) < d$ [31, 32]. Furthermore, $\chi_2(PG(8)) = \chi_2(PG(16)) = 3$ and $\chi_2(PG(32)) = 4$ [32].
- It is known from [13, Theorem 8] that $\chi_2(PG(64)) = 5$. Consequently, $\chi_2(PG(128)) \geq \chi_2(PG(64)) = 5$.
- $\chi_2(PG(2^d)) \rightarrow \infty$ for $d \rightarrow \infty$ by [17, Corollary 2].
- The 2-chromatic number $\chi_2(PG(8) \cup PG(8)^{P_3}) = 3$ by direct computer calculations.
- The 2-chromatic number $\chi_2(PG(16) \cup PG(16)^{P_4}) = 3$ by direct computer calculations.
- The 2-chromatic number $\chi_2(PG(32) \cup PG(32)^{P_5}) = 4$ since $\chi_2(PG(32)) \geq 4$ and it is possible with a computer to find a $(4, 2)$ -coloring of this genus 126 nonorientable surface on 31 vertices.
- The 2-chromatic number $\chi_2(PG(128) \cup PG(128)^{P_7})$ equals 5 or 6 since $\chi_2(PG(128)) \geq 5$, and it is possible with a computer to find a $(6, 2)$ -coloring of this genus 2542 nonorientable surface on 127 vertices with $f = (127, 8001, 5334)$. This is the smallest known example (in terms of number of faces) of a combinatorial surface with 2-chromatic number at least 5. (For an orientable genus 620 surface with $f = (2017, 9765, 6510)$ and $\chi_2 = 5$ see Section 2.) A list of facets $PG128_PG128P_7$ of the triangulation $PG(128) \cup PG(128)^{P_7}$ can be found online at [3].

Theorem 5.8. *The 2-chromatic number of the genus 2542 nonorientable combinatorial surface $PG(128) \cup PG(128)^{P_7}$ on 127 vertices is 5 or 6.*

The authors of [15] assert on p. 333 that $PG(32)$ has a ‘cyclic bi-embedding in an orientable surface’. We have not been able to verify this assertion. However, we go along with Theorem 5 in [27] that rules out orientable bi-embeddings of $PG(2^d)$ for $5 \leq d \leq 19$. Presumably, by Corollary 3.19.(3), $GL(5, \mathbf{F}_2) \cap GL(5, \mathbf{F}_2)^T$ would contain a Sylow 31-subgroup of $GL(5, \mathbf{F}_2)$ for any such transversal T , but this contradicts Proposition 5.6 according to which such transversals are nonorientable.

It is claimed in [12, Theorem 3.1] that $PG(N)$ admits an (orientable?) transversal for all $N = 2^d$. The proof appears to be incorrect.

6. CHROMATIC NUMBERS OF HIGHER-DIMENSIONAL MANIFOLDS

It was noted in the introduction that the chromatic numbers of a (compact) triangulable d -manifold M^d form a descending sequence

$$\chi_1(M^d) \geq \chi_2(M^d) \geq \cdots \geq \chi_d(M^d) \geq \chi_{d+1}(M^d) = 1.$$

We shall now see that roughly the first half of these chromatic numbers are infinite. As usual, B^d denotes the d -dimensional ball and S^d the d -dimensional sphere. Then $\chi_s(M^d) \geq \chi_s(B^d)$ for $s \leq d$ and $\chi_s(S^d) = \chi_s(B^d)$ for $s < d$.

Proposition 6.1. $\chi_s(S^d) \leq \chi_s(S^{d+1})$ for $d \geq 1$ and $1 \leq s \leq d$.

Proof. Any triangulation of S^d is a subcomplex of a triangulation of its suspension S^{d+1} . \square

Proposition 6.2. $\chi_s(S^d) = \infty$ for $d \geq 3$ and $1 \leq s < \lceil d/2 \rceil$.

Proof. Let $d \geq 3$ be odd. Then S^d admits a triangulation $\partial\text{CP}(m, d+1)$ as the boundary of the cyclic polytope $\text{CP}(m, d+1)$ on m vertices in \mathbf{R}^{d+1} . The triangulation $\text{CP}(m, d+1)$ is $\lfloor (d+1)/2 \rfloor$ -neighborly in the sense that it has the same s -skeleton as the full $(m-1)$ -simplex Δ^{m-1} when $s < \lfloor \frac{d+1}{2} \rfloor = \lceil d/2 \rceil$. Thus $\text{CP}(m, d+1)$ has the same s -chromatic number, $\lceil m/s \rceil$, as Δ^{m-1} when $1 \leq s < \frac{d+1}{2}$. Since we can choose m to be arbitrarily large, and since $\chi_s(S^{d-1}) \leq \chi_s(S^d)$ for even d by Lemma 6.1, the lemma follows. \square

Proposition 6.3. $\chi_s(S^d) = \infty$ when $d \geq 3$ and $1 \leq s \leq \lceil d/2 \rceil$.

Proof. Let $s = \lceil d/2 \rceil$ for $d \geq 3$. Euclidean $(2s-1)$ -space contains s -dimensional geometric simplicial complexes with arbitrarily high s -chromatic numbers [18, Theorem 22]. Since we can extend these embedded triangulations to triangulations of B^{2s-1} [6, Theorem I.2.A], we see that $\chi_s(S^{2s-1}) = \infty$. Then also $\chi_s(S^{2s}) = \infty$ by the monotonicity of Lemma 6.1. \square

Theorem 6.4. $\chi_s(M^d) = \infty$ when M^d is a triangulable d -manifold with $d \geq 3$ and $s \leq \lceil d/2 \rceil$.

Proof. Follows from Lemma 6.3 by taking connected sums of a triangulation M^d with triangulations of S^d . \square

The interesting chromatic numbers of a d -manifold M^d of dimension $d \geq 3$ are

$$\chi_s(M^d), \quad \lceil d/2 \rceil < s \leq d.$$

In particular, for a manifold M of dimension 3, the only unknown chromatic number is $\chi_3(M)$. The most basic question here is to determine $\chi_3(S^3)$. More generally, it remains an open problem to determine $\chi_d(S^d)$ as a function of the dimension $d \geq 3$.

7. AN EXPLICIT EXAMPLE OF A COMBINATORIAL 3-SPHERE WITH $\chi_2 = 5$

Combinatorial 3-spheres can have arbitrarily large 2-chromatic number (Lemma 6.3). However, it seems to be hard to construct or find explicit examples with $\chi_2 > 4$.

The first place to look for concrete examples certainly are boundary complexes of cyclic 4-polytopes. Yet, their 2-chromatic numbers are at most 3 even though they have arbitrarily large 1-chromatic numbers.

Example 7.1. The boundaries $\partial\text{CP}(m, 4)$ of cyclic 4-polytopes on $m \geq 4$ vertices are neighborly combinatorial 3-spheres with $\chi_1(\partial\text{CP}(m, 4)) = m$, but

$$\chi_2(\partial\text{CP}(m, 4)) = \begin{cases} 2, & m \text{ even,} \\ 3, & m \text{ odd.} \end{cases}$$

As before for surfaces in Section 2, we tried bistellar flips [7] to search through the space of triangulations of S^3 , but never found an example with $\chi_2 > 4$. Also, we have not been able to find in the literature a single concrete example of a combinatorial 3-sphere not admitting a $(4, 2)$ -coloring. At least, the literature does contain sporadic examples of combinatorial 3-spheres with 2-chromatic number equal to 4.

Example 7.2. Altshuler's ‘peculiar’ combinatorial 3-sphere [1] with $f = (10, 45, 70, 35)$ has 2-chromatic number $\chi_2 = 4$ (as we verified with the computer).

One reason for why it is hard to find triangulations of S^3 with $\chi_2 > 4$ is that by the previous sections we do not have a small example of a triangulated orientable surface with $\chi_2 > 4$. Thus, if we search for obstructions to $(4, 2)$ -colorings, triangulated orientable surfaces that are common as subcomplexes of S^3 will not help.

In [18], Heise, Panagiotou, Pikhurko and Taraz provided an inductive geometric construction (this construction was also found independently by Jan Kynčl and Josef Cibulka as pointed out to us by Martin Tancer) to yield 2-dimensional complexes with arbitrary high 2-chromatic numbers. The basic idea of the construction is as follows. Let $T(r-1)$ be a geometric 2-dimensional simplicial complex in \mathbf{R}^3 that has all its vertices on the moment curve (t, t^2, t^3) and that has $f = (f_0, f_1, f_2)$ and $\chi_2(T(r)) \geq r-1$. Then we can obtain from $T(r-1)$ a new geometric

2-dimensional simplicial complex $T(r)$ in \mathbf{R}^3 that again has all its vertices on the moment curve, but now has $f = \left(\binom{r}{2}f_0 + r, \binom{r}{2}(f_1 + 1 + 2f_0), \binom{r}{2}(f_0 + f_2)\right)$ and $\chi_2(T(r)) \geq r$.

As an abstract simplicial complex, $T(r)$ is obtained from $T(r-1)$ by taking $\binom{r}{2}$ copies of $T(r-1)$ ‘attached’ to the $\binom{r}{2}$ edges of a complete graph K_r : For every edge e of K_r and the corresponding copy of $T(r-1)$ a triangle ev is added to the complex $T(r)$ for each vertex v of $T(r-1)$.

If we take for $T(2)$ a single triangle with $\chi_2 = 2$, then

- $f(T(2)) = (3, 3, 1)$ and $\chi_2(T(2)) = 2$,
- $f(T(3)) = (12, 30, 12)$ and $\chi_3(T(2)) = 3$,
- $f(T(4)) = (76, 330, 144)$ and $\chi_3(T(2)) = 4$,
- $f(T(5)) = (765, 4830, 2200)$ and $\chi_3(T(2)) = 5$.

By choosing the vertices of $T(r)$ appropriately on the moment curve [18], $T(r)$ is realized as a geometric 2-dimensional simplicial complex in \mathbf{R}^3 . According to [6, Theorem I.2.A] (we are grateful to Karim Adiprasito for reminding us of this result from PL topology), we can extend $T(r)$ to a triangulation of the 3-dimensional ball B^3 , which contains $T(r)$ as a subcomplex and thus has $\chi_2(T(r)) \geq r$. However, the resulting triangulations of B^3 will be of tremendous size.

In the following, we give a first concrete example of a non-(4, 2)-colorable combinatorial 3-sphere by using a topological version of the geometric construction of [18].

Theorem 7.3. *There is a non-(4, 2)-colorable 3-sphere `non_4_2_colorable` with face vector $f = (167, 1579, 2824, 1412)$.*

Proof. The basic idea for constructing a ‘small’ example of a non-(4, 2)-colorable 3-sphere is as follows. We embed a complete graph K_5 in 3-space and attach tetrahedra to it

- so that eventually K_5 is embedded in a triangulated 3-ball (with 166 vertices)
- in a way that the attached tetrahedra prevent the K_5 to have a monochromatic edge.

We then add to the constructed ball the cone over its boundary to close the triangulation to a non-(4, 2)-colorable 3-sphere (with 167 vertices). Indeed, if we can guarantee that in any 2-coloring of the resulting triangulation none of the 10 edges of the complete graph K_5 is allowed to be monochromatic (in four colors), then all admissible 2-colorings of the triangulation must have at least five colors.

To each of the 10 edges of K_5 we will attach a (small) 15-vertex ball B_{15} with the properties that

- B_{15} has a (4, 2)-coloring, but *no* (3, 2)-coloring,
- B_{15} has all its 15 vertices on its boundary
- and the 15 vertices can be lined up to form a Hamiltonian path on the boundary.

A ball with these properties is obtained from the 16-vertex `double_trefoil` sphere $S_{16,92}$ of [4] by deleting the star of the vertex 7 and renaming vertex 16 to 7. The list of facets of the ball B_{15} is given in Table 1.

TABLE 1. The ball B_{15} .

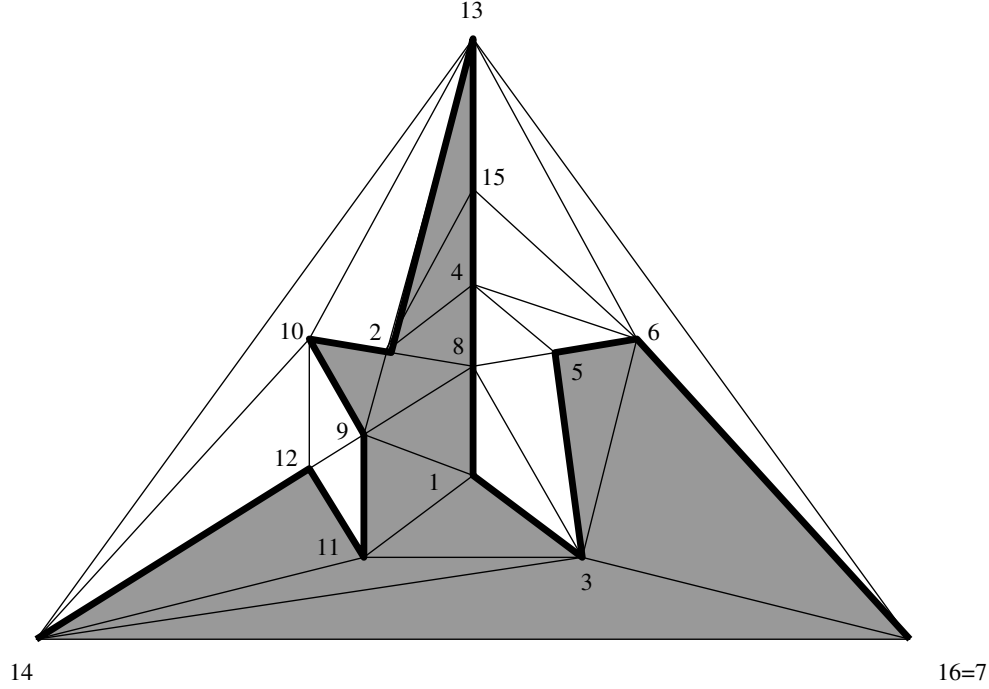
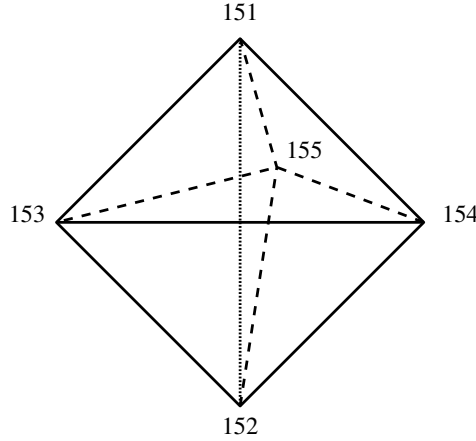
1 2 5 6	1 2 5 12	1 2 6 12	1 3 8 11	1 4 5 6	1 4 5 16/7	1 4 6 12	1 4 10 13
1 4 10 16/7	1 4 12 13	1 5 12 13	1 5 13 16/7	1 8 9 14	1 8 10 14	1 8 10 15	1 8 11 15
1 9 11 15	1 9 14 15	1 10 13 14	1 10 15 16/7	1 13 14 16/7	1 14 15 16/7	2 3 4 13	2 3 4 15
2 3 13 15	2 4 8 16/7	2 4 10 13	2 4 10 16/7	2 5 6 14	2 5 12 14	2 6 8 12	2 6 8 16/7
2 6 9 14	2 6 9 16/7	2 8 9 14	2 8 12 14	2 9 10 16/7	3 4 12 13	3 4 12 15	3 5 6 14
3 5 8 11	3 5 11 14	3 6 9 14	3 6 9 16/7	3 9 12 13	3 9 12 16/7	3 9 13 15	3 9 14 15
3 12 15 16/7	3 14 15 16/7	4 5 8 16/7	4 6 12 15	5 8 11 13	5 8 13 16/7	5 11 12 13	5 11 12 14
6 8 12 15	6 8 13 15	6 8 13 16/7	8 10 12 14	8 10 12 15	8 11 13 15	9 10 12 16/7	9 11 12 13
9 11 13 15	10 12 15 16/7						

It is easy to check (by an exhaustive computer search) that B_{15} has no (3, 2)-coloring. For the boundary of B_{15} and the Hamiltonian path 14–12–11–9–10–2–13–15–4–8–1–3–5–6–7 (in bold) on the boundary see Figure 9. In total, our construction of a non-(4, 2)-colorable 3-sphere will have 167 vertices,

- 5 vertices for the complete graph K_5 (vertices 151–155),
- 10×15 vertices for the 10 copies of the 15-vertex ball B_{15} (vertices 1–150),
- 10 vertices to glue the 10 balls to the 10 edges of K_5 (vertices 156–165),
- 1 vertex to close an inner hole (vertex 166)
- and 1 vertex to cone over the boundary to close the triangulation (vertex 167).

The construction of the 3-sphere `non_4_2_colorable` is in six steps.

Step I. We embed the complete graph K_5 in 3-space as the 1-skeleton of a bipyramid over a triangle 153 154 155, where, in addition, we connect the two apices 151 and 152 by an edge 151–152, as depicted in Figure 10.

FIGURE 9. The boundary of the 3-ball B_{15} .FIGURE 10. The complete graph K_5 in 3-space.

Step II. We add the six triangles of the bipyramid and also two interior tetrahedra 151 152 153 155 and 151 152 154 155. This way, we obtain a mixed 2- and 3-dimensional complex that has a single tetrahedral cavity 151 152 153 154. (The two interior tetrahedra and the empty tetrahedron are aligned around the vertical central edge 151–152.)

Step III. To the nine edges of K_5 that lie on the boundary of the bipyramid, the copies of B_{15} can be attached so that the attached balls point ‘outwards’, while for the central edge 151–152 the attached copy points into the cavity. For the attaching itself, we add a chain of 14 tetrahedra, where each of the tetrahedra is composed by the join of an edge of K_5 and an edge of the Hamiltonian path on the boundary (of a copy) of B_{15} . In Figure 11, we depict the attaching (via the Hamiltonian path) of the original copy of B_{15} to the edge 151–152. *It is at this point that we ensure that the respective K_5 -edge cannot be monochromatic in one of four colors:* Since B_{15} has no $(3, 2)$ -coloring, we are forced to use four(!) colors to color the vertices of the ball. Each of the vertices of B_{15} appears as a vertex v of the Hamiltonian path on the boundary and is connected to the respective K_5 -edge, say, 151–152, by a triangle v 151 152 in the chain of tetrahedra. It follows that the edge 151–152 cannot be monochromatic in one of four colors, since otherwise one of the triangles in the chain would be monochromatic, which is not allowed.

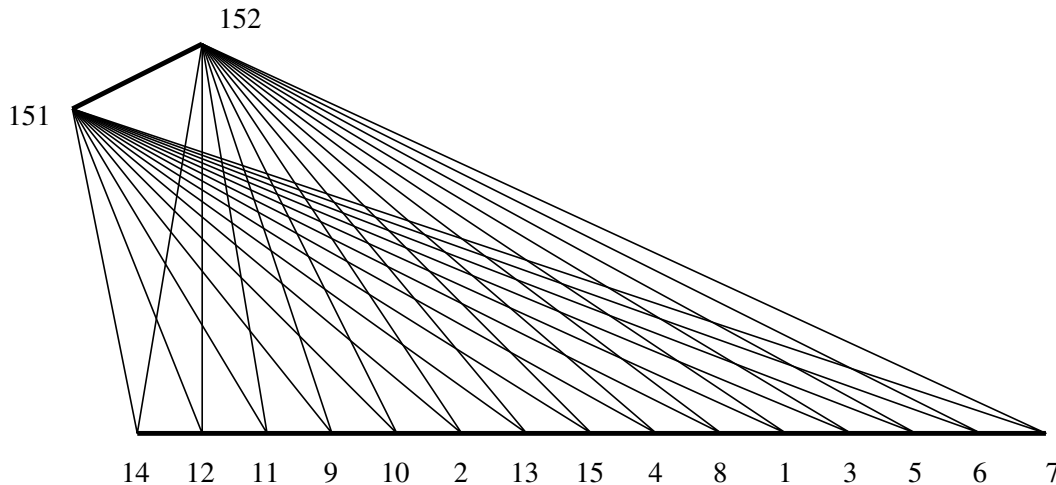


FIGURE 11. The chain of 14 tetrahedra to attach (a copy of) the ball B_{15} to an edge of K_5 .

Step IV. At this point, we have constructed a mixed 2- and 3- dimensional simplicial complex that is embedded in 3-space, but which is not $(4, 2)$ -colorable. This complex has one cavity that we are going to fill in Step V. However, we first thicken the attaching ridges (via the Hamiltonian paths) of the 10 balls. For this, we add to each copy of the ball the cone with respect to a new vertex over the grey shaded triangles of Figure 9 and, in addition, over the triangles of one of the sides of the chain of tetrahedra of Figure 11, respectively.

Step V. We fill the cavity by adding the cone with respect to a new vertex 166 over the triangles that enclose the cavity. Now, we have obtained a non- $(4, 2)$ -colorable 3-ball with 166 vertices.

Step VI. We add the cone over the boundary of the ball to close the triangulation to a non- $(4, 2)$ -colorable 3-sphere with $f = (167, 1579, 2824, 1412)$. \square

Corollary 7.4. *The 2-chromatic number of the example `non_4_2_colorable` is 5.*

Proof. A $(5, 2)$ -coloring of the non- $(4, 2)$ -colorable 3-sphere `non_4_2_colorable` was found with the computer; a list of facets of the example is available online at [3]. \square

Remark 7.5. The example `non_4_2_colorable` has turned out to be of interest for testing heuristics for the computations of discrete Morse vectors [5]. It has perfect discrete Morse vector $(1, 0, 0, 1)$, but this vector is hard to find due to the 10 knotted balls that are used in the construction.

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